

# HIGHER CYCLIC OPERADS

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**ABSTRACT.** We introduce a convenient definition for higher cyclic operads, which is based on unrooted trees and Segal conditions. More specifically, we introduce a category  $\Xi$  of trees, which carries a tight relationship to the Moerdijk-Weiss category of rooted trees  $\Omega$ . We prove a nerve theorem exhibiting colored cyclic operads as presheaves on  $\Xi$  which satisfy a Segal condition. Finally, we introduce a Quillen model category whose fibrant objects satisfy a weak Segal condition, and we consider these objects as an up-to-homotopy generalization of the concept of cyclic operad.

For certain operads, such as the moduli space of Riemann spheres with labeled punctures or the endomorphism operad of a vector space  $V$  equipped with a non-degenerate bilinear form, there is not really a qualitative difference between the notion of input and output. Indeed, in the former case, the ‘output’ of a given element arises solely from our choice of labels and not the underlying geometry, while in the latter case we have natural isomorphisms

$$\mathrm{End}_V(n) = \mathrm{hom}(V^{\otimes n}, V) = \mathrm{hom}(V^{\otimes n}, V^*) = \mathrm{hom}(V^{\otimes n+1}, k).$$

This consideration leads directly to the notion of *cyclic operad*, introduced by Getzler and Kapranov in [21] (although we add the axiom due to van der Laan, see [29, §11] and [33, §II.5.1]). A cyclic operad is an operad  $O$  with extra structure, namely an action of the cyclic group  $C_{n+1} = \langle \tau \rangle$  on the space  $O(n)$ . Applied to an element  $f \in O(n)$ , we should regard  $f\tau \in O(n)$  as  $f$  with the first input changed to the output and the output changed to the last input, as in figure 1. To get a feel for how this cyclic operator should act on compositions, one should look at trees with several vertices like the one in in figure 2. Given a cyclic operad, one can begin talking about graph homology (see [28] as well as the generalizations of Conant

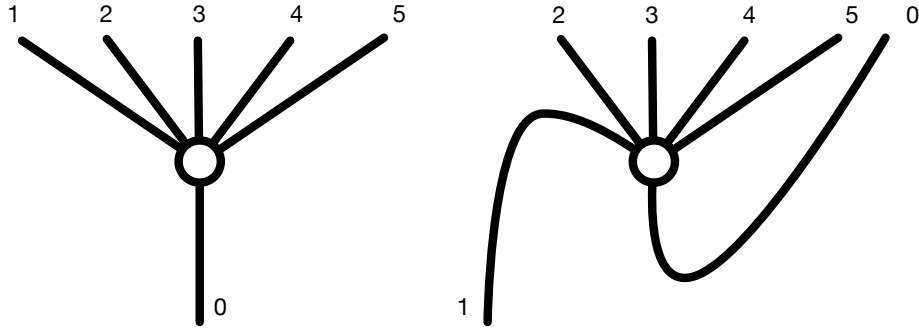
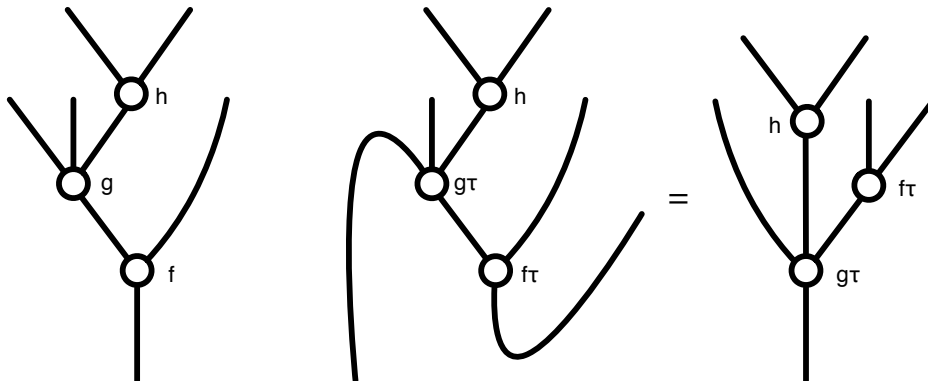


FIGURE 1.  $f$  and  $f\tau$

FIGURE 2. A composition of  $f, g, h$ , and the action of  $\tau$  on this composition

& Vogtmann [18]), whereas algebras over cyclic operads admit a cyclic homology theory [21].

Further examples of cyclic operads include the associative, Lie, and commutative operads, (certain models for) the framed little  $n$ -disks operad [12, 26], the  $A_\infty$  operad [21], and also any monoid with involution [30, 38] (that is, the involution  $x \mapsto x^\dagger$  satisfies  $x^\dagger y^\dagger = (yx)^\dagger$ ) regarded as an operad concentrated in degree 1. The last of these is useful for giving small examples (see example 5.8 and proposition 9.10), but also gives a connection with another interesting class of mathematical objects.

A dagger category [37, 2.2] is a category  $\mathcal{C}$  together with an involutive functor  $\dagger : \mathcal{C}^{op} \rightarrow \mathcal{C}$  which is the identity on objects. In other words, a dagger structure on  $\mathcal{C}$  is an assignment  $(f : X \rightarrow Y) \mapsto (f^\dagger : Y \rightarrow X)$  satisfying  $f^{\dagger\dagger} = f$  and  $f^\dagger g^\dagger = (gf)^\dagger$ ; this is the many-objects version of a monoid with involution. As Baez argued eloquently in [1], to understand the similarities between general relativity and quantum theory, one should begin by considering the natural dagger structure on the category of  $n$ -cobordisms and on the category of Hilbert spaces, respectively. Other important examples of dagger categories include any groupoid, categories of relations, and categories of correspondences.

Colored cyclic operads [25] are a simultaneous generalization of cyclic operads (which we might term ‘monochrome cyclic operads’) and of dagger categories. There are additional examples in the literature (eg [17, 5.4] [39, 3.11.1]), and the concept provides a bridge to approaches to higher operads based on colored operads.

The higher cyclic operads that we develop in this paper are a variation on ‘dendroidal models’ for  $\infty$ -operads (cf [9, 14, 15, 16, 35]). The dendroidal category  $\Omega$  is a category of rooted trees [34]; each such rooted tree  $T$  can be regarded as a free object in the category of  $\text{Ed}(T)$ -colored operads. The dendroidal category is then defined to be the full subcategory of the category of all colored operads whose objects are the rooted trees. Not only is  $\Omega$  defined as a subcategory of colored operads, but it turns out that there is a model structure on the category of presheaves of  $\Omega$  that is Quillen equivalent to a model structure on the category of simplicially-enriched colored operads [14, 15, 16]. This is an extension of the

equivalence between the Joyal model structure on simplicial sets and the Bergner model structure on simplicially-enriched categories (see [8] for references).

It would be a very ambitious project to attempt to do all of the above for colored cyclic operads, and we are skeptical that the full Cisinski-Moerdijk program can be carried out in the cyclic case. A key sticking point is that the adjunction between categories and dagger categories is badly behaved, in particular with respect to equivalences. Thus, in the present paper we limit what is said about colored cyclic operads. It is true that every unrooted tree  $S$  freely generates an  $\text{Ed}(S)$ -colored cyclic operad  $C(S)$  (see section 6), but we do not ever consider the full-subcategory of colored cyclic operads spanned by the unrooted trees. Instead, we directly construct a category  $\Xi$  of unrooted trees that is reminiscent of  $\Omega$ . The assignment  $S \mapsto C(S)$  gives a faithful, non-full functor from  $\Xi$  to **Cyc** (theorem 6.4, example 6.5). We use this to prove a nerve theorem for colored cyclic operads (theorem 7.4, lemma 7.2).

Our main goal is to propose a model for weak monochrome cyclic operads. These are called *Segal cyclic operads* in section 9, and they are certain reduced presheaves satisfying a Segal condition. The Segal cyclic operads are patterned after the Segal operads appearing in the work of Bergner and the first author [9], which have become important in current work of Boavida, Horel, and the second author on profinite completions of the framed little disks operad.

The profinite completion of a product of spaces is weakly equivalent, but in general not isomorphic, to the product of the profinite completions. For this reason, the profinite completion of an operad does not yield an operad on the nose but rather an  $\infty$ -operad. This fact has played a crucial role in work of Horel [24] when he generalized work of Fresse [20] and computed a profinite version of the Grothendieck-Teichmüller group  $\widehat{GT} \cong \pi_0 \text{End}^h(\widehat{D_2})$ , where  $D_2$  is the little 2-disks operad. In forthcoming work by Boavida, Horel and the second author, they show that considering the framed little 2 disks as an operad, they recover exactly the same result, ie  $\widehat{GT} \cong \pi_0 \text{End}^h(\widehat{D_2}) \cong \pi_0 \text{End}^h(f\widehat{D_2})$ . Considering  $fD_2$  as a cyclic operad would necessarily result in a smaller set of endomorphisms and conjecturally would provide refinement on these computations; of course one would expect the profinite completion of a cyclic operad to be some type of infinity cyclic operad. Providing a good foundation for this project is one of the major motivations for the present paper.

**Overview.** We give a brief outline of the paper. Each section begins with a more substantial summary of its contents.

The first section is dedicated to the construction of the category  $\Xi$  of unrooted trees. In the second section, we examine exactly how close  $\Xi$  is to the category  $\Omega$  of rooted trees. The third and fourth sections are devoted to two structures on the category  $\Xi$ : a generalized Reedy structure and an active / inert (or generic / free) weak factorization system.

The next three sections deal with the relationship of  $\Xi$  to colored cyclic operads. In the fifth section give a definition for colored cyclic operads and fix notation, while in the sixth we construct the functor  $\Xi \rightarrow \mathbf{Cyc}$ . Finally, in the seventh section, we prove a nerve theorem for colored cyclic operads.

The final two sections are devoted to model-categorical matters. The penultimate section is about the model structure on diagrams indexed by a generalized

Reedy category, and at the beginning of this section we show that this model structure usually has properties which ensure that Bousfield localizations exist. We then restrict to the case when the base category is the category of simplicial sets. In section 8.1 we discuss certain cases when categories of *reduced* presheaves of simplicial sets admit model structures. In section 8.2 we show that these model categories are in fact simplicial model categories.

In the last section we prove the existence of a model structure on reduced  $\Xi$ -presheaves in simplicial sets whose fibrant objects, the Segal cyclic operads, satisfy a Segal condition. We show that there is a Quillen adjunction (which is not a Quillen equivalence) between this model structure and the model structure for Segal operads from [9].

**Notational conventions.** If  $\mathcal{C}$  is a category, we will write  $\mathcal{C}(x, y)$  or  $\text{hom}(x, y)$  for the set of morphisms from  $x$  to  $y$ , depending on if the name of our category is short (eg  $\mathcal{C} = \Xi$ ) or long (eg  $\mathcal{C} = \mathbf{sSet}_*^{\Xi^{op}}$ ). We will write  $\text{Iso}_{\mathcal{C}}(x, y)$  for the isomorphisms from  $x$  to  $y$ ,  $\text{Aut}_{\mathcal{C}}(x) := \text{Iso}_{\mathcal{C}}(x, x)$  for the invertible self-maps of  $x$ , and  $\text{Iso}(\mathcal{C})$  for the wide subcategory of  $\mathcal{C}$  consisting of all of the isomorphisms. In all adjunctions  $\mathcal{C} \rightleftarrows \mathcal{D}$ , the top arrow denotes the left adjoint.

Throughout this paper we use freely the language of Quillen model categories and take the book of Hirschhorn [23] as our standard reference.

## 1. THE UNROOTED TREE CATEGORY $\Xi$

The main goal of this section is to define a category of unrooted trees  $\Xi$ . We will begin with a formalism for general graphs, before defining the objects of  $\Xi$  in definition 1.3. The morphisms are presented in definition 1.12. We then embark on a sustained study of the nature of these morphisms; key tools are the notions of distances and (minimal) paths in trees. Along the way, we recover the Moerdijk-Weiss dendroidal category  $\Omega$ . Finally, in theorem 1.33, we show that  $\Xi$  is actually a category.

At the heart of this work is the notion of ‘graph with legs’. One can choose several formalisms; for concreteness, let us say that an *undirected graph with legs* consists of two finite sets  $E$  and  $V$  and a function  $\text{Nbhd} : V \rightarrow \mathcal{P}(E)$  (the set of subsets of  $E$ ). This data should satisfy one axiom, namely that, for each  $e \in E$

$$|\{v \in V \mid e \in \text{Nbhd}(v)\}| \leq 2.$$

We will package the triple  $(E, V, \text{Nbhd})$  into a single symbol  $G$ , and write  $\text{Ed}(G) = E$  and  $\text{Vt}(G) = V$ . Edges actually come in two types, namely interior edges

$$\text{Int}(G) = \{e \in E \mid e \in \text{Nbhd}(v) \cap \text{Nbhd}(w) \text{ for some } v \neq w\}$$

and the set of legs

$$\text{Legs}(G) = \text{Ed}(G) \setminus \text{Int}(G)$$

which are edges incident to at most one edge. If  $v$  is a vertex of  $G$ , we also write  $|v|$  for the valence of  $v$ , or the cardinality of the set  $\text{Nbhd}(v)$ .

Every graph has an underlying topological space, which can be described as follows. See the left hand side of figure 3 for an example.

**Definition 1.1** (Space associated to a graph). Fix an  $\epsilon$  with  $0 < \epsilon < 1$ , which we can use to scale the closed unit disc  $\mathbb{D}$  in the complex plane  $\mathbb{C}$ . Define

$$|\star_0| = \epsilon\mathbb{D} = \{re^{i\theta} \mid 0 \leq r \leq \epsilon, 0 \leq \theta \leq 2\pi\} \subsetneq \mathbb{D} \subsetneq \mathbb{C}$$

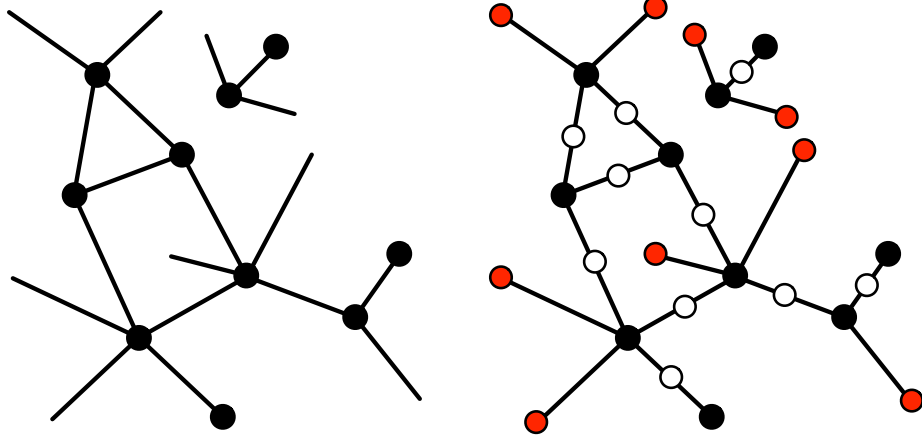


FIGURE 3. A graph with legs and its corresponding rwb graph

and, for  $n > 0$ ,

$$|\star_n| = \epsilon\mathbb{D} \cup \bigcup_{k=0}^{n-1} \{re^{\frac{k}{n}2\pi i} \mid 0 \leq r \leq 1\},$$

considered as a subspace in the closed unit disc of the complex plane. If  $G$  is an undirected graph with legs, fix bijections

$$\kappa_v : \text{Nbhd}(v) \xrightarrow{\cong} \{e^{\frac{k}{|v|}2\pi i}\} = S^1 \cap |\star_{|v|}|$$

and define  $|G| = [0, 1] \times \text{Ed}(G)$  when  $\text{Vt}(G) = \emptyset$  and otherwise

$$|G| = \frac{\prod_{v \in \text{Vt}(G)} |\star_{|v|}|}{\kappa_v(e) \sim \kappa_w(e)}$$

where  $e \in \text{Nbhd}(v) \cap \text{Nbhd}(w)$ .

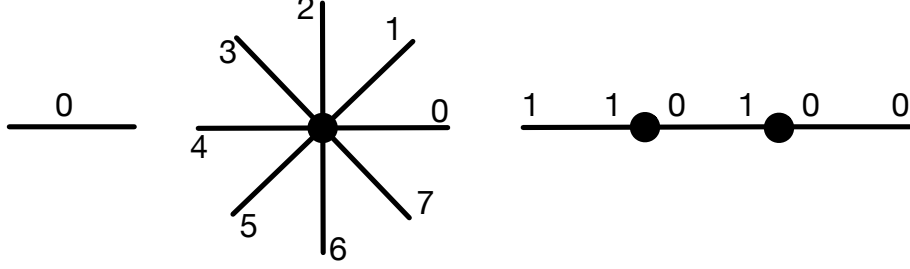
Notice that the homeomorphism type of  $|G|$  determines the isomorphism type of  $G$ . This would not be the case if we did not add some thickness at the centers of  $|\star_n|$  by using the  $\epsilon\mathbb{D}$ . Indeed, a variation of realization with  $\epsilon = 0$  produces the closed unit interval  $[0, 1]$  on both the graph  $G_1$  with one edge and no vertices to the graph  $G_2$  with one vertex  $v$ , one edge  $e$ , and  $\text{Nbhd}(v) = \{e\}$ .

The following is an alternative, equivalent formalism for graph with legs.

**Definition 1.2** (Red / white / black formalism). A *rwg graph* is an ordinary undirected graph (see, for example [19, §1.1]) where each vertex is colored either red, white, or black and such that

- red vertices are univalent,
- white vertices are bivalent and are only adjacent to black vertices, and
- a black vertex is not adjacent to any other black vertex.

From a graph with legs, we can form a rwg graph by coloring all vertices black, adding a white vertex on each interior edge, and adding a red vertex to the loose end of each leg. Each rwg graph determines a graph with legs by deleting the white

FIGURE 4. The trees  $\eta = L_0$ ,  $\star_8$ , and  $L_2$ .

vertices and joining the edges on either side and deleting all of the red vertices. See figure 3 for an illustration of this correspondence.

**1.1. Trees.** The category  $\Xi$  governing cyclic dendroidal sets has ‘unrooted’ or ‘cyclic’ trees as objects.

**Definition 1.3.** An *unpinned tree*  $G$  is an undirected graph with legs which is contractible, has at least one leg, and is equipped with bijections

$$\text{ord}^v : \{0, 1, \dots, n_v\} \xrightarrow{\cong} \text{Nbhd}(v),$$

where  $\text{Nbhd}(v) \subseteq \text{Ed}(G)$  is the set of vertices adjacent to  $v$ . A *pinned tree*, or just *tree*, has, in addition, a bijection

$$\text{ord} : \{0, 1, \dots, n\} \xrightarrow{\cong} \text{Legs}(G),$$

where  $\text{Legs}(G) \subseteq \text{Ed}(G)$  is the set of legs of  $G$ .

**Example 1.4.** Let us fix several foundational examples of trees (figure 4).

- The graph with one edge and no vertices, which we write as  $\eta$ .
- For each  $n > 0$ , the graph  $\star_n$ . This graph has a single vertex  $v$  and  $n$  edges  $\{0, 1, \dots, n-1\}$  (and take  $\text{ord}^v = \text{ord} = \text{id}$ ).<sup>1</sup>
- For  $n \geq 0$ , linear graphs  $L_n$  with

$$\begin{aligned} \text{Vt}(L_n) &= \{v_1, \dots, v_n\} & \text{Ed}(L_n) &= \{e_0, \dots, e_n\} \\ \text{Nbhd}(v_i) &= \{e_{i-1}, e_i\} & \text{Legs}(L_n) &= \{e_0, e_n\} \end{aligned}$$

with  $\text{ord}^{v_i}(t) = e_{i-1+t}$ ,  $\text{ord}(0) = e_0$ , and  $\text{ord}(1) = e_n$ . Note that  $L_0 = \eta$ .

- We will call any tree with all vertices bivalent a linear graph.

**Remark 1.5.** A variation on [19, 1.5.2] shows that we can build up any (unpinned) tree by iteratively attaching a  $\star_k$  at a leg. This shows that the space  $|G|$  inherits the structure of a metric space induced from those on  $|\star_k|$ ; one checks that this is not dependent on the order of attaching. Further, this gives a metric structure on  $\text{Vt}(G) \amalg \text{Ed}(G)$  by regarding this as a subset of  $|G|$  via  $v \mapsto 0_v = 0 \in |\star_v| \subseteq |G|$  and  $e$  adjacent to  $v$  goes to  $\kappa_v(e) \in |\star_v| \subseteq |G|$ .

<sup>1</sup>Note the shift in index compared with [33, p. 250]: they use the notation  $*_n$  for what we call  $\star_{n+1}$ .

**Remark 1.6.** A directed tree is a tree where each edge has an orientation; another way to say this when  $G \neq \eta$  is to say that there are partitions

$$\text{Nbhd}(v) = \text{out}(v) \amalg \text{in}(v)$$

$$\text{Legs}(G) = \text{out}(G) \amalg \text{in}(G)$$

so that  $\text{out}(v) \cap \text{out}(w) = \emptyset = \text{in}(v) \cap \text{in}(w)$  for  $v \neq w$  and  $\text{out}(v) \cap \text{in}(G) = \emptyset = \text{in}(v) \cap \text{out}(G)$ . This description actually has a little bit more information floating around than we would like; namely a  $(|\text{in}(G)|, |\text{out}(G)|)$ -shuffle and, for each  $v \in \text{Vt}(G)$ , a  $(|\text{in}(v)|, |\text{out}(v)|)$ -shuffle. This is simply because for a directed graph we only need separate orderings on the inputs and outputs, not orderings on the entire neighborhoods.

Making a choice for the  $(p, q)$ -shuffles above, every directed tree determines a tree. We will use the convention that the total order on  $\text{Nbhd}(v)$  is determined by that on  $\text{out}(v)$  and  $\text{in}(v)$  by insisting that for  $e \in \text{out}(v)$  and  $e' \in \text{in}(v)$ , that  $e < e'$ . Similarly to get an order on  $\text{Legs}(G)$  by saying  $\text{out}(G) < \text{in}(G)$  (unless  $G$  consists of a single edge). This convention makes it so that for a rooted tree, the downward edge is always labeled by '0'. This gives the map  $\text{Ob}(\Omega) \rightarrow \text{Ob}(\Xi)$ ; we will actually define a variant of  $\Omega$  in 1.19.

**1.2. Morphisms of  $\Xi$ .** When discussing subgraphs of trees, we will always assume that they are nonempty, connected, and contain all edges incident to any of their vertices.

**Definition 1.7.** An *subgraph* of  $S$  consists of a pair of subsets

$$V \subseteq \text{Vt}(S)$$

$$E \subseteq \text{Ed}(S)$$

so that

- if  $v \in V$ , then  $\text{Nbhd}(v) \subseteq E$  (which means that  $R = (V, E, \text{Nbhd})$  constitutes the structure of an undirected graph without orderings),
- the underlying space of the graph  $R = (V, E, \text{Nbhd})$  is contractible.

Write  $\text{Sbgph}(S)$  for the set of subgraphs of  $S$ .

**Remark 1.8.** Subgraphs of  $S$  are naturally *unpinned* trees. The vertex orderings  $\text{ord}^v$  are inherited from those in  $S$ .

**Example 1.9.** Each edge  $e \in S$  constitutes a subgraph with  $E = \{e\}$  and  $V = \emptyset$ . We will write this subgraph as  $|_e$ .

**Example 1.10.** For each  $v \in \text{Vt}(G)$ , there is a subgraph  $\star_v$  with  $V = \{v\}$  and  $E = \text{Nbhd}(v)$ . Thus we have an inclusion  $\star : \text{Vt}(G) \hookrightarrow \text{Sbgph}(G)$ . Notice that  $\star_v$  has a preferred ordering with

$$\text{ord}_{\star_v}^v = \text{ord}_{\star_v} = \text{ord}_G^v : \{0, 1, \dots, n\} \xrightarrow{\cong} \text{Nbhd}(v) = \text{Legs}(\star_v).$$

**Proposition 1.11.** If  $R$  and  $R'$  are subgraphs of  $S$  and  $R \cap R' \neq \emptyset$ , then  $R \cup R'$  is also a subgraph of  $S$ .

*Proof.* Write  $R = (V, E)$  and  $R' = (V', E')$ . The first condition we need to check for  $R \cup R' = (V \cup V', E \cup E')$  is immediate, and does not require the hypothesis. The hypothesis  $R \cap R' \neq \emptyset$  means  $(V \cap V') \cup (E \cap E') \neq \emptyset$ , which implies that the underlying space of  $R \cup R'$  is connected (since it is the union of the underlying

spaces of  $R$  and  $R'$ ). Thus it is a connected subspace of a contractible graph, hence is contractible as well.  $\square$

**Definition 1.12.** Suppose that  $R$  and  $S$  are trees.

- A *morphism*  $\phi : R \rightarrow S$  is defined to be a pair of maps

$$\begin{aligned}\phi_0 : \text{Ed}(R) &\rightarrow \text{Ed}(S) \\ \phi_1 : \text{Vt}(R) &\rightarrow \text{Sbgph}(S)\end{aligned}$$

satisfying the following:

- (1) If  $v$  is not bivalent (that is,  $|\text{Nbhd}(v)| \neq 2$ ), then  $\phi_0|_{\text{Nbhd}(v)}$  is injective.
- (2) For each vertex  $v$ ,  $\phi_0(\text{Nbhd}(v)) = \text{Legs}(\phi_1(v))$  (as unordered sets).
- (3)  $\text{Vt}(\phi_1(v)) \cap \text{Vt}(\phi_1(w)) = \emptyset$  for  $v \neq w$ .
- The *identity map*  $\text{id}_R : R \rightarrow R$  is given by letting  $(\text{id}_R)_0 = \text{id}_{\text{Ed}(R)}$  and letting  $(\text{id}_R)_1$  be the inclusion  $\star : \text{Vt}(R) \rightarrow \text{Sbgph}(R)$ .
- More generally, a morphism  $\phi : R \rightarrow S$  is an *isomorphism* if  $\phi_0$  is a bijection and if  $\phi_1$  factors through  $\text{Vt}(S)$  as  $\phi_1 = \star \tilde{\phi}_1$  with  $\tilde{\phi}_1$  a bijection.

$$\begin{array}{ccc} \text{Vt}(R) & \xrightarrow{\phi_1} & \text{Sbgph}(S) \\ & \searrow \tilde{\phi}_1 \cong & \uparrow \star \\ & & \text{Vt}(S) \end{array}$$

We will describe composition<sup>2</sup> later (definition 1.29), after which the reader may wish to verify that this definition of isomorphism is correct from a categorical standpoint.

**Example 1.13.** Suppose that  $R = R'$  except for orderings. There is a unique isomorphism  $\phi : R \rightarrow R'$  with  $\phi_0 = \text{id}$ . The map  $\phi_1$  is just the inclusion

$$\star : \text{Vt}(R) \hookrightarrow \text{Sbgph}(R) = \text{Sbgph}(R').$$

To see that  $\phi$  is unique, induct on the number of vertices  $n$  of  $R$ ; the case  $n = 0$  is clear since there is only one map  $\eta \rightarrow \eta$ . Suppose that uniqueness has been established for all  $m < n$ ; pick any  $e_0 \in \text{Legs}(R)$ . There is a unique  $v_0 \in \text{Vt}(R)$  with  $e_0 \in \text{Nbhd}(v_0)$ ; By assumption  $\phi_0(e_0) = e_0$ , hence  $\phi_1(v_0) = \star_{v_0}$ . For each  $e \in \text{Nbhd}(v_0) \setminus e_0$ , there is a subgraph  $R_e$  of  $R$  consisting of all vertices and edges on all paths not containing  $v$  but beginning at  $e$ . By the induction hypothesis,  $\phi|_{R_e}$  is uniquely determined by the fact that it is the identity on edges. Hence  $\phi$  is uniquely determined as well.

**Remark 1.14.** The argument for uniqueness in the previous example fails if we allow graphs without legs. Indeed, the graph  $\bullet \bullet$  admits two distinct automorphisms  $\phi$  with  $\phi_0 = \text{id}$ .

A *path* in a graph  $G$  is an alternating word in the alphabet  $\text{Ed}(G) \sqcup \text{Vt}(G)$  which can only contain a subword  $ve$  or  $ev$  if  $e \in \text{Nbhd}(v)$ . A path from a vertex  $v$  to a vertex  $w$  is a path of the form

$$P = ve_1v_1e_2 \dots e_{n-1}v_{n-1}e_nw$$

<sup>2</sup>As Ben Ward pointed out to us, one could instead define a morphism  $R \rightarrow S$  as a pair  $\text{Ed}(R) \rightarrow \text{Ed}(S)$  and  $\text{Vt}(R)_+ \leftarrow \text{Vt}(S)$  going in opposite directions (with a modified set of axioms), which makes composition more clear.



while a path from an edge  $e$  to an edge  $e'$  is a path of the form

$$P = ev_1e_1v_2 \dots v_{n-1}e_{n-1}v_ne'.$$

The length of a path is the length of the word.

We can concatenate paths  $P$  and  $P'$  if the last letter in  $P$  is the first letter in  $P'$  or if the last letter in  $P$  is adjacent to the last letter in  $P'$ . In the former case, we will remove the duplicate letter. This takes the form:

$P$	$P'$	$PP'$
$\dots ev$	$ve' \dots$	$\dots eve' \dots$
$\dots ev$	$e'v' \dots$	$\dots eve'v' \dots$
$\dots ve$	$ev' \dots$	$\dots vev' \dots$
$\dots ve$	$v'e' \dots$	$\dots vev'e' \dots$

**Definition 1.15.** Suppose that  $v, w \in \text{Vt}(G)$ ,  $e, f \in \text{Ed}(G)$ .

- The distance from  $v$  to  $w$  is the number of edges  $n$  appearing in the shortest path  $P = ve_1v_1e_2 \dots e_{n-1}v_{n-1}e_nv$  from  $v$  to  $w$ ; that is,

$$d(v, w) = \min_P \frac{|P| - 1}{2},$$

where  $P$  ranges over all paths from  $v$  to  $w$ .

- Likewise, the distance from  $e$  to  $f$  is

$$d(e, f) = \min_P \frac{|P| - 1}{2}$$

where  $P$  ranges over all paths from  $e$  to  $f$ .

The  $\frac{1}{2}$  scaling factor is not particularly important; if  $\tilde{d}$  is the induced metric (see remark 1.5) on  $|G|$ , we have  $d(e, f) = \frac{1}{2}\tilde{d}(\kappa_{v_e}(e), \kappa_{v_f}(f))$  and  $d(v, w) = \frac{1}{2}\tilde{d}(0_v, 0_w)$ , where  $e \in \text{Nbhd}(v_e)$  and  $0_v = 0 \in \star_{|v|}$  (and likewise for  $f$  and  $w$ ). Alternatively, if we consider the rwb graph  $G_{rwb}$  associated to  $G$ , then  $\text{Vt}(G)$  is the same as the set of black vertices of  $G_{rwb}$  and  $\text{Ed}(G)$  is the union of the sets of red and white vertices of  $G_{rwb}$ ; these distances are then half of the usual graph distance between vertices (see, eg [19, §1.3]) in  $G_{rwb}$ .

**Definition 1.16.** Let  $G$  be a tree. If  $v, w \in \text{Vt}(G)$ , then a *minimal path* from  $v$  to  $w$  is a path  $P$  from  $v$  to  $w$  so that  $d(v, w) = (|P| - 1)/2$ . Likewise, if  $e, f \in \text{Ed}(G)$ , then a *minimal path* from  $e$  to  $f$  is a path  $P$  from  $e$  to  $f$  so that  $d(e, f) = (|P| - 1)/2$ .

**Proposition 1.17.** *Let  $G$  be a tree. If  $v, w \in \text{Vt}(G)$ , then a minimal path from  $v$  to  $w$  exists and is unique. A similar statement applies to minimal paths between edges. Minimal paths are characterized as those containing no repeated entries.*

*Proof.* Every vertex and edge of  $G$  is a vertex in the corresponding rwb graph  $G_{rwb}$ . This statement is then [19, Theorem 1.5.1] applied to the tree  $G_{rwb}$ .  $\square$

Existence and uniqueness of minimal paths gives the following:

**Corollary 1.18.** *If  $R$  is a subgraph of  $S$ , then the distance in  $R$  between any two vertices (resp. edges) of  $R$  is the same as the distance in  $S$  between those vertices (resp. edges).*  $\square$

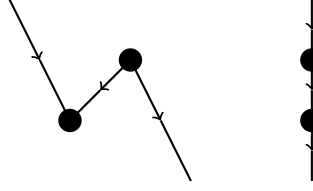


FIGURE 5. Two different directed structures on the same undirected graph

We now have the necessary tools to define the objects and morphisms of the dendroidal category  $\Omega$ . Though we do not make serious use of  $\Omega$  until section 2, we include this definition here, rather than after theorem 1.33, to indicate the usefulness of the notion of distance.

**Definition 1.19** (Dendroidal category). We now define (a variant of)  $\Omega$  as a subcategory of  $\Xi$ .

- A *rooted tree* is a tree  $R$  satisfying the following condition: Suppose  $r_0 = \text{ord}(0) \in \text{Legs}(R)$ . If  $v \in \text{Vt}(R)$  and  $k > 0$ , then

$$d(\text{ord}^v(0), r_0) < d(\text{ord}^v(k), r_0).$$

In this case, we set

$$\begin{aligned} \text{in}(v) &= \text{Nbhd}(v) \setminus \text{ord}^v(0) & \text{out}(v) &= \text{ord}^v(0) \\ \text{in}(R) &= \text{Legs}(R) \setminus \text{ord}(0) & \text{out}(R) &= \text{ord}(0) = r_0. \end{aligned}$$

- If  $R$  and  $S$  are rooted trees and  $\phi : R \rightarrow S$  is a map in  $\Xi$ , we say that  $\phi$  is *oriented* if for each  $v \in \text{Vt}(R)$  and each  $k > 0$ ,

$$d(\phi_0(\text{ord}^v(0)), s_0) \leq d(\phi_0(\text{ord}^v(k)), s_0).$$

- The category  $\Omega$  is the subcategory of  $\Xi$  (which is a category by theorem 1.33) whose objects are rooted trees and whose morphisms are the oriented maps between rooted trees.
- We write  $\iota : \Omega \rightarrow \Xi$  for the subcategory inclusion.

Notice that if  $\phi$  is an oriented map, then  $\phi_1(v)$  is a rooted tree (without ordering of the leaves) with root  $\phi_0(\text{ord}^v(0)) = \phi_0(\text{out}(v))$ .

**Remark 1.20.** This definition of  $\Omega$  is analogous to that of  $\Omega'$  in [3, Example 2.8]. Our rooted trees are equivalent to a rooted tree together with a planar structure and an ordering of the input edges, and morphisms do not need to preserve the planar structure.

**Remark 1.21.** In the above definition we were able to *recognize* rooted trees among all trees; we cannot do something similar for general directed trees (and hence for the category  $\Theta$  from [22, 6.55]). Indeed, graphs which are linear as undirected graphs generally possess many directed structures, even controlling for the number of inputs and outputs. See Figure 5. In short, there is a functor from (a legged-variant of)  $\Theta$  to  $\Xi$ , but it is not injective on objects.

**Lemma 1.22.** Let  $\phi : R \rightarrow S$  be a morphism of  $\Xi$ . Then  $\text{Int}(\phi_1(v)) \cap \text{Ed}(\phi_1(w)) = \emptyset$  if  $v \neq w$

*Proof.* Suppose that  $\phi_1(w)$  contains a vertex and  $e \in \text{Int}(\phi_1(v)) \cap \text{Ed}(\phi_1(w))$ . Since  $\phi_1(w) \neq |_e$ , there is a vertex  $w' \in \text{Vt}(\phi_1(w))$  with  $e \in \text{Nbhd}(w')$ . Since  $e \in \text{Int}(\phi_1(v))$ , we know that  $w' \in \text{Vt}(\phi_1(v))$ , which implies  $v = w$  by 1.12(3).

In the general case, we induct on  $d(v, w)$ . If  $d(v, w) = 0$ , then  $v = w$  and there is nothing to show. Assume the statement is true whenever  $d(v, w) < n$ . Suppose we have vertices  $v$  and  $w$  with  $d(v, w) = n > 0$  and let  $ve_1v_1e_2v_2 \dots e_{n-1}v_{n-1}e_nw$  be the shortest path from  $v$  to  $w$ . If  $\phi_1(w)$  contains a vertex then we know  $\text{Int}(\phi_1(v)) \cap \text{Ed}(\phi_1(w)) = \emptyset$  by the first paragraph. Suppose that  $\phi_1(w) = |_e$  is a single edge. Then  $e = \phi_0(e_n) \in \text{Legs}(\phi_1(v_{n-1}))$ , which implies that  $e \notin \text{Int}(\phi_1(v))$  by the induction hypothesis.  $\square$

**Lemma 1.23.** *Let  $\phi : R \rightarrow S$  be a morphism of  $\Xi$ . For each vertex  $v \in \text{Vt}(R)$ ,  $\text{Int}(\phi_1(v)) \subseteq \text{Ed}(S) \setminus \text{Im}(\phi_0)$ .*

*Proof.* Suppose that  $\phi_0(e) \in \text{Int}(\phi_1(v))$ . Since the graph has a vertex  $v$  and is connected, every edge is adjacent to at least one vertex. If  $e$  is adjacent to  $v$ , then  $\phi_0(e) \in \phi_0(\text{Nbhd}(v)) = \text{Legs}(\phi_1(v)) \subset \text{Ed}(S) \setminus \text{Int}(\phi_1(v))$ , so we conclude that  $e$  is not adjacent to  $v$ . Thus there exists a  $w \neq v$  with  $e \in \text{Nbhd}(w)$ . But now

$$\phi_0(e) \in \text{Int}(\phi_1(v)) \cap \text{Legs}(\phi_1(w)) \subset \text{Int}(\phi_1(v)) \cap \text{Ed}(\phi_1(w)),$$

which is empty by lemma 1.22.  $\square$

**Lemma 1.24.** *Let  $\phi : R \rightarrow S$  be a morphism of  $\Xi$ . If*

$$|\text{Legs}(\phi_1(v)) \cap \text{Legs}(\phi_1(w))| > 1,$$

*then  $v = w$ .*

*Proof.* Suppose  $v \neq w$ . Let  $e, e' \in \text{Legs}(\phi_1(v)) \cap \text{Legs}(\phi_1(w))$ . Let  $P$  be the shortest path in  $\phi_1(v)$  from  $e$  to  $e'$  and let  $P'$  be the shortest path in  $\phi_1(w)$  from  $e$  to  $e'$ . Since  $P, P'$  contain no repeated entries by proposition 1.17, they are also distance minimizing paths in  $S$ . By uniqueness,  $P = P'$ . If  $e \neq e'$ , this path contains a vertex, hence  $\emptyset \neq \text{Vt}(\phi_1(v)) \cap \text{Vt}(\phi_1(w))$  and we see that  $v = w$  by definition 1.12(3).  $\square$

**Lemma 1.25.** *Suppose that  $\phi : R \rightarrow S$  is a morphism of  $\Xi$ . If  $\phi_0(e) = \phi_0(e')$ , then  $e$  and  $e'$  lie on a common linear subgraph (which may just mean  $e = e'$ ), all of whose edges map to a common value.*

*Proof.* Induct on  $d(e, e')$ . If  $d(e, e') = 0$  then  $e = e'$  and the result follows. If  $d(e, e') = 1$  and  $\phi_0(e) = \phi_0(e')$ , then the vertex adjacent to both  $e$  and  $e'$  must be bivalent. Assume the result is known for  $d(e, e') < n$ . Suppose  $\phi_0(e) = \tilde{e} = \phi_0(e')$  with  $d(e, e') = n > 1$ . Let  $e_0v_1e_1v_2 \dots v_{n-1}e_{n-1}v_ne_n$  be the distance minimizing path in  $R$  from  $e = e_0$  to  $e' = e_n$ .

For each  $i$ , let  $P_i$  be the shortest path in  $\phi_1(v_i)$  from  $\phi_0(e_{i-1})$  to  $\phi_0(e_i)$ . The path  $P_1 \dots P_{n-1}$  contains no repeated entries by lemma 1.22 and definition 1.12(3), hence is the unique length minimizing path (proposition 1.17) from  $\tilde{e}$  to  $\phi_0(e_{n-1})$ . Both of these edges are in  $\phi_1(v_n)$ , hence  $P_1 \dots P_{n-1}$  is a path in  $\phi_1(v_n)$ . If  $\tilde{e} \neq \phi_0(e_{n-1})$ , then  $P_1 \dots P_{n-1}$  contains a vertex, violating definition 1.12(3). Thus  $\phi_0(e_n) = \tilde{e} = \phi_0(e_{n-1})$ , so  $v_n$  is bivalent. The result now follows from induction since  $d(e, e_{n-1}) < d(e, e')$ .  $\square$

We will momentarily (in 1.27) define the *image* of a map, which is essentially the union of all of the subgraphs  $\phi_1(v)$ . We first check that this union actually is a subgraph.

**Proposition 1.26.** *Suppose that  $\phi : R \rightarrow S$  is in  $\Xi$  and  $R \neq \eta$ . Then*

$$\bigcup_{v \in \text{Vt}(R)} \phi_1(v)$$

*is a subgraph of  $S$ .*

*Proof.* Suppose that  $R$  contains a vertex. Let  $P = v_1 e_1 \dots v_{n-1} e_{n-1} v_n$  be a path in  $R$  containing all vertices at least once. Then we have

$$\text{Im}(\phi) = \phi_1(v_1) \cup \phi_1(v_2) \cup \dots \cup \phi_1(v_n).$$

Use induction. By lemma 1.11 we know that

$$\left( \phi_1(v_1) \cup \dots \cup \phi_1(v_k) \right) \cup \phi_1(v_{k+1})$$

is a subgraph since  $\phi_1(v_1) \cup \dots \cup \phi_1(v_k)$  and  $\phi_1(v_{k+1})$  are (induction hypothesis) and

$$\phi_0(e_k) \in \phi_1(v_k) \cap \phi_1(v_{k+1}).$$

□

**Definition 1.27.** Let  $\phi : R \rightarrow S$  be a map in  $\Xi$ . Define the *image* of  $\phi$ , denoted  $\text{Im}(\phi)$ , to be the subgraph

$$\text{Im}(\phi) = \begin{cases} |\phi_0(e) & R = \eta \\ \bigcup_{v \in \text{Vt}(R)} \phi_1(v) & \text{Vt}(R) \neq \emptyset \end{cases}$$

of  $S$ .

**Proposition 1.28.** *Suppose that  $\phi : R \rightarrow S$  is in  $\Xi$ . Then  $\phi_0(\text{Legs}(R)) = \text{Legs}(\text{Im}(\phi))$ .*

*Proof.* The desired identity is clear when  $R = \eta$  is an edge. We show that  $\phi_0(\text{Legs}(R)) = \text{Legs}(\text{Im}(\phi))$  when  $R$  contains a vertex.

To show that  $\text{Legs}(\text{Im}(\phi)) \subset \phi_0(\text{Legs}(R))$ , we prove the equivalent statement: if  $f \in \text{Im}(\phi)$  and  $\phi_0^{-1}(f) \subset \text{Int}(R)$ , then  $f \in \text{Int}(\text{Im}(\phi))$ .

- If  $|\phi_0^{-1}(f)| = 0$ , then there exists a  $v$  with  $f \in \text{Int}(\phi_1(v)) \subset \text{Int}(\text{Im}(\phi))$ .
- If  $\phi_0^{-1}(f) = \{e\} \subset \text{Int}(R)$ , then there are  $w_1 \neq w_2$  with  $e \in \text{Nbhd}(w_1) \cap \text{Nbhd}(w_2)$ . If  $\phi_1(w_i)$  is an edge, then  $|\phi_0^{-1}| > 1$ . Thus each of  $\phi_1(w_1)$  and  $\phi_1(w_2)$  contains a vertex, hence a vertex adjacent to  $f$ . It follows that  $f$  is an interior edge of  $\text{Im}(\phi)$ .
- If  $|\phi_0^{-1}(f)| \geq 2$ , let  $e, e'$  be elements of greatest distance within the set  $\phi_0^{-1}(f)$ ; note that  $e \neq e'$ . Let  $e_0 v_1 \dots e_{n-1} v_n e_n$  be the minimal path from  $e = e_0$  to  $e' = e_n$ , where  $n \geq 1$ . Since  $e, e'$  are internal edges, there are vertices  $w \neq v_1$  and  $w' \neq v_n$  so that  $w e_0 v_1 \dots e_{n-1} v_n e_n w'$  is a well-defined path. By proposition 1.17,  $w \neq w'$ . By maximality of  $d(e, e')$ , each of  $\phi_1(w)$  and  $\phi_1(w')$  contain at least one vertex. Hence  $f$  is adjacent to a vertex  $v$  in  $\phi_1(w)$  and a vertex  $v' \in \phi_1(w')$ . Since  $\text{Vt}(\phi_1(w)) \cap \text{Vt}(\phi_1(w')) = \emptyset$ , we know that  $v \neq v'$ , hence  $f \in \text{Int}(\text{Im}(\phi))$ .

In order to prove the reverse inclusion, we show  $\text{Int}(\text{Im}(\phi)) \cap \phi_0(\text{Legs}(R)) = \emptyset$ . We already know that  $\text{Int}(\phi_1(v)) \cap \phi_0(\text{Ed}(R)) = \emptyset$  for each vertex  $v$  by lemma 1.23. Suppose that

$$f \in \text{Int}(\text{Im}(\phi)) \setminus \left( \bigcup_{v \in \text{Vt}(R)} \text{Int}(\phi_1(v)) \right) = \text{Int}(\text{Im}(\phi)) \cap \phi_0(\text{Ed}(R)).$$

Then there exist  $w_1 \neq w_2$ ,  $v_1 \neq v_2$ , with  $w_i \in \phi_1(v_i)$  and  $\{f\} = \text{Nbhd}(w_1) \cap \text{Nbhd}(w_2)$ .

- If  $|\phi_0^{-1}(f)| = 1$ , then  $\phi_0^{-1}(f) = \{e\} = \text{Nbhd}(v_1) \cap \text{Nbhd}(v_2)$ . Since  $e \in \text{Int}(R)$ ,  $\phi_0(e) = f \notin \phi_0(\text{Legs}(R))$ .
- Suppose  $|\phi_0^{-1}(f)| > 1$ ; we know by lemma 1.25 that all elements of  $\phi_0^{-1}(f)$  lie on a common linear subgraph  $L$ . Let  $e$  and  $e'$  be the extremities of  $L$ ; the minimal path from  $e$  to  $e'$  is of the form  $e_0 \tilde{v}_1 e_1 \dots e_{n-1} \tilde{v}_n e_n$  with  $e_0 = e$  and  $e_n = e'$ . Since there are no repeated elements in this minimal path, we know that  $e_1, \dots, e_{n-1} \in \text{Int}(R)$ . Each  $\phi_1(\tilde{v}_i)$  is just the single edge  $|_f$ , while  $\phi_1(v_i)$  contains the vertex  $w_i$ . Hence  $v_1, v_2 \notin \{\tilde{v}_1, \dots, \tilde{v}_n\} = \text{Vt}(L)$ , and we see that  $e, e' \in \text{Int}(R)$ . We have thus shown that  $\text{Ed}(L) = \phi_0^{-1}(f) \subset \text{Int}(R)$ , thus  $f \notin \phi_0(\text{Legs}(R))$ .

□

**Definition 1.29.** Let  $\phi : R \rightarrow S$  and  $\psi : S \rightarrow T$  be morphisms of  $\Xi$ . Define two functions

$$\begin{aligned} (\psi \circ \phi)_0 : \text{Ed}(R) &\rightarrow \text{Ed}(T) \\ (\psi \circ \phi)_1 : \text{Vt}(R) &\rightarrow \text{Sbgph}(T) \end{aligned}$$

by  $(\psi \circ \phi)_0 = \psi_0 \circ \phi_0$  and

$$(\psi \circ \phi)_1(v) = \text{Im}(\psi|_{\phi_1(v)}).$$

**Lemma 1.30.** If  $R$  is a subgraph of  $S$ ,  $i : R \rightarrow S$  is the inclusion, and  $\psi : S \rightarrow T$ , then  $\psi|_R = \psi \circ i$  is a map of  $\Xi$ .

*Proof.* We show that  $\psi|_R$ , specified by  $(\psi|_R)_0 = \psi_0|_{\text{Ed}(R)}$  and  $(\psi|_R)_1 = \psi_1|_{\text{Vt}(R)}$  satisfies the three conditions of definition 1.12. Since valency of vertices is preserved by subgraph inclusion, (1) holds. For (2), we have, for  $v \in \text{Vt}(R)$ ,

$$(\psi|_R)_0(\text{Nbhd}(v)) = \psi_0(\text{Nbhd}(v)) \stackrel{(2)}{=} \text{Legs}(\psi_1(v)) = \text{Legs}((\psi|_R)_1(v)).$$

Finally, (3) is established by

$$\text{Vt}((\psi|_R)_1(v)) \cap \text{Vt}((\psi|_R)_1(w)) = \text{Vt}(\psi_1(v)) \cap \text{Vt}(\psi_1(w)) = \emptyset$$

when  $v \neq w$  are in  $\text{Vt}(R)$ . □

**Proposition 1.31.** If  $\phi : R \rightarrow S$  and  $\psi : S \rightarrow T$  are morphisms of  $\Xi$ , then so is  $\psi \circ \phi$ .

*Proof.* Let us first address (1) of definition 1.12. Suppose that  $e \neq e' \in \text{Nbhd}(v)$ . We wish to show either that  $(\psi \circ \phi)_0(e) \neq (\psi \circ \phi)_0(e')$  or that  $v$  is bivalent. If  $(\psi \circ \phi)_0(e) = (\psi \circ \phi)_0(e')$ , then  $\phi_0(e)$  and  $\phi_0(e')$  lie on a common linear subgraph by lemma 1.25, which implies that  $v$  was bivalent.

We turn to (2). Choose an ordering for  $\text{Legs}(\phi_1(v))$  to make  $\phi_1(v)$  into a pinned tree, that is, an object of  $\Xi$ . Let  $i : \phi_1(v) \rightarrow S$  be the subgraph inclusion. By lemma 1.30, we know  $\psi|_{\phi_1(v)}$  is a map in  $\Xi$ . We then have

$$\begin{aligned} \text{Legs}((\psi \circ \phi)_1(v)) &= \text{Legs}(\text{Im}(\psi|_{\phi_1(v)})) \\ &\stackrel{1.28}{=} (\psi|_{\phi_1(v)})_0(\text{Legs}(\phi_1(v))) \\ &= (\psi|_{\phi_1(v)})_0\phi_0(\text{Nbhd}(v)) \\ &= (\psi \circ \phi)_0(\text{Nbhd}(v)). \end{aligned}$$

For the last requirement (3), suppose that  $p$  is in the set

$$\begin{aligned} &\text{Vt}((\psi \circ \phi)_1(v)) \cap \text{Vt}((\psi \circ \phi)_1(w)) \\ &= \text{Vt}(\text{Im}(\psi|_{\phi_1(v)})) \cap \text{Vt}(\text{Im}(\psi|_{\phi_1(w)})) \\ &= \left( \bigcup_{x \in \text{Vt}(\phi_1(v))} \text{Vt}(\psi_1(x)) \right) \cap \left( \bigcup_{y \in \text{Vt}(\phi_1(w))} \text{Vt}(\psi_1(y)) \right). \end{aligned}$$

Then  $p \in \text{Vt}(\psi_1(x_0)) \cap \text{Vt}(\psi_1(y_0))$  for some  $x_0 \in \text{Vt}(\phi_1(v))$ ,  $y_0 \in \text{Vt}(\phi_1(w))$ , hence  $x_0 = y_0$  by (3) applied to  $\psi$ . Then  $x_0 = y_0 \in \phi_1(v) \cap \phi_1(w)$ , so  $v = w$  by (3) applied to  $\phi$ .  $\square$

**Proposition 1.32.** *The composition from definition 1.29 is associative and unital.*

*Proof.* For unitality, suppose that  $\phi : R \rightarrow S$  is in  $\Xi$ , and recall the definition of  $\text{id}_R$  from 1.12. We have  $(\text{id}_S \circ \phi)_0 = \phi_0 = (\phi \circ \text{id}_R)_0$ . For any  $v \in R$ , we have

$$(\text{id}_S \circ \phi)_1(v) = \text{Im}(\text{id}_S|_{\phi_1(v)}) = \phi_1(v)$$

and

$$(\phi \circ \text{id}_R)_1(v) = \text{Im}(\phi|_{(\text{id}_R)_1(v)}) = \phi_1(v).$$

Hence  $\text{id}_S \circ \phi = \phi = \phi \circ \text{id}_R$ .

Suppose that  $\phi : R \rightarrow S$ ,  $\psi : S \rightarrow T$ , and  $\chi : T \rightarrow U$  are morphisms of  $\Xi$ . We will show that

$$(\chi \circ \psi) \circ \phi = \chi \circ (\psi \circ \phi).$$

We have

$$[(\chi \circ \psi) \circ \phi]_0 = (\chi_0 \circ \psi_0) \circ \phi_0 = \chi_0 \circ (\psi_0 \circ \phi_0) = [\chi \circ (\psi \circ \phi)]_0.$$

Suppose  $v \in \text{Vt}(R)$ . If  $(\psi \circ \phi)_1(v)$  is an edge  $e \in \text{Ed}(T)$ , then  $\phi_1(v)$  is a linear graph and  $\psi(w) = |_e$  for every  $w \in \phi_1(v)$ . In this case, we have

$$[\chi \circ (\psi \circ \phi)]_1(v) = \text{Im}(\chi|_{(\psi \circ \phi)_1(v)}) = |_{\chi_0(e)}$$

and

$$\begin{aligned} [(\chi \circ \psi) \circ \phi]_1(v) &= \text{Im}((\chi \circ \psi)|_{\phi_1(v)}) \\ &= \begin{cases} |_{\chi_0\psi_0(e')} = |_{\chi_0(e)} & \phi_1(v) \text{ is an edge } e' \\ \bigcup_{w \in \phi_1(v)} (\chi \circ \psi)_1(w) & \text{Vt}(\phi_1(v)) \neq \emptyset \end{cases} \\ &= \begin{cases} |_{\chi_0(e)} & \phi_1(v) \text{ is an edge} \\ \bigcup_{w \in \phi_1(v)} \text{Im}(\chi|_{\psi_1(w)}) & \text{Vt}(\phi_1(v)) \neq \emptyset \end{cases} \\ &= |_{\chi_0(e)} = [\chi \circ (\psi \circ \phi)]_1(v). \end{aligned}$$

Suppose that  $\text{Vt}((\psi \circ \phi)_1(v)) \neq \emptyset$ . Then  $\phi_1(v)$  has a vertex as well. We have

$$\begin{aligned}
[\chi \circ (\psi \circ \phi)]_1(v) &= \text{Im}(\chi|_{(\psi \circ \phi)_1(v)}) \\
&= \text{Im}(\chi|_{\text{Im}(\psi|_{\phi_1(v)})}) \\
&= \bigcup_{m \in \text{Vt}(\text{Im}(\psi|_{\phi_1(v)}))} \chi_1(m) \\
&= \bigcup_{\substack{m \in \bigcup_{t \in \phi_1(v)} \psi_1(t)}} \chi_1(m) \\
&= \bigcup_{t \in \phi_1(v)} \bigcup_{m \in \psi_1(t)} \chi_1(m).
\end{aligned}
\quad
\begin{aligned}
[(\chi \circ \psi) \circ \phi]_1(v) &= \text{Im}((\chi \circ \psi)|_{\phi_1(v)}) \\
&= \bigcup_{w \in \phi_1(v)} (\chi \circ \psi)_1(w) \\
&= \bigcup_{w \in \phi_1(v)} \text{Im}(\chi|_{\psi_1(w)}) \\
&= \bigcup_{w \in \phi_1(v)} \bigcup_{x \in \psi_1(w)} \chi_1(x)
\end{aligned}$$

hence  $[(\chi \circ \psi) \circ \phi]_1(v) = [\chi \circ (\psi \circ \phi)]_1(v)$ .  $\square$

Let us now summarize our work from this section.

**Theorem 1.33.** *With the objects from definition 1.3, the morphisms and identities from definition 1.12, and the composition from definition 1.29,  $\Xi$  is a category.*  $\square$

## 2. USING ROOTING TO ORIENT MAPS IN $\Xi$

In this section we give a careful comparison of the morphism sets of  $\Xi$  and  $\Omega$ . Morally, the category  $\Xi$  is built up from the dendroidal category  $\Omega$  by adding isomorphisms which rotate trees. Thus, every morphism of  $\Xi$  should decompose into an oriented map (in  $\Omega$ ) along with some rotation data.

In the present section we make this precise. To each tree  $S$  and a choice of root  $s_0 \in \text{Ed}(S)$ , there is a rooted tree  $\mathcal{T}(S, s_0) \in \Omega$  (see definition 2.1). Further, given a morphism  $\phi : R \rightarrow S$  we can transform  $\phi$  into an oriented map (lemma 2.9)

$$\mathcal{L}(\phi) : \mathcal{T}(R, r_\phi) \rightarrow \mathcal{T}(S, s_0)$$

for some particular choice of root  $r_\phi \in \text{Legs}(R)$  (see definition 2.4). We show that  $\mathcal{L}$  respects composition in a certain sense (proposition 2.11). Finally, in theorem 2.12 (see also corollary 2.15 and proposition 2.16) we realize our goal and make clear the idea that (non-constant) maps  $R \rightarrow S$  are just certain maps in  $\Omega$  along with rooting data for  $S$ .

**Definition 2.1** (Rooting of trees). Suppose we are given a pair  $(S, s_0)$  with  $S \in \text{Ob}(\Xi)$  and  $s_0 \in \text{Ed}(S)$ .

- We now define a rooted tree  $T \in \text{Ob}(\Omega)$  with  $\text{Ed}(T) = \text{Ed}(S)$  and  $\text{Vt}(T) = \text{Vt}(S)$ . For each  $v \in \text{Vt}(S)$ , let  $\text{ord}^v(k_v) = \text{out}(v) \in \text{Nbhd}(v)$  be the element which minimizes the function  $d(-, s_0)|_{\text{Nbhd}(v)}$ . Setting  $\text{in}(v) = \text{Nbhd}(v) \setminus \text{out}(v)$ , we have an induced ordering on  $\text{in}(v)$  via

$$\begin{array}{ccc}
\{0, 1, \dots, n_v\} & \xrightarrow[\cong]{\text{ord}^v} & \text{Nbhd}(v) \\
\uparrow +k_v \mod (n_v+1) & & \parallel \\
\{0, 1, \dots, n_v\} & & \{\text{out}(v)\} \amalg \text{in}(v) \\
\uparrow & & \uparrow \\
\{1, \dots, n_v\} & \xrightarrow[\cong]{\dots\dots\dots} & \text{in}(v).
\end{array}$$

Similarly, we have an induced ordering on  $\text{in}(T) = \text{Legs}(S) \setminus s_0$  via

$$\begin{array}{ccc}
 \{0, 1, \dots, n\} & \xrightarrow[\cong]{\text{ord}} & \text{Legs}(S) \\
 \uparrow +k \bmod (n+1) & & \parallel \\
 \{0, 1, \dots, n\} & & \{s_0\} \amalg \text{in}(T) \\
 \uparrow & & \uparrow \\
 \{1, \dots, n\} & \xrightarrow[\cong]{} & \text{in}(T),
 \end{array}$$

where  $k = \text{ord}^{-1}(s_0)$ . Although  $\iota T \neq S$ , since they have different total orderings (though the same *cyclic* orderings)  $\text{ord}$  and  $\text{ord}^v$ , there is a *unique* isomorphism  $f : \iota T \rightarrow S$  of  $\Xi$  with  $f_0 = \text{id}$  by example 1.13.

- We write

$$\mathcal{T}(S, s_0) = (T, f : \iota T \xrightarrow{\cong} S)$$

for this construction. The second component is redundant (since we insist that  $f_0 = \text{id}$ ), so we will usually abuse notation and just write  $\mathcal{T}(S, s_0) = T$ .

- Define a map  $\mathcal{A}_{s_0}$  as the composite

$$\begin{array}{ccc}
 \coprod_{r \in \text{Legs}(R)} \Omega(\mathcal{T}(R, r), \mathcal{T}(S, s_0)) & \xrightarrow{\mathcal{A}_{s_0}} & \Xi(R, S) \\
 \downarrow \iota & & \uparrow \cong \Xi(\text{id}, f_{s_0}) \\
 \coprod_{r \in \text{Legs}(R)} \Xi(\iota \mathcal{T}(R, r), \iota \mathcal{T}(S, s_0)) & \xrightarrow{\coprod_r \Xi(f_r^{-1}, \text{id})} & \Xi(R, \iota \mathcal{T}(S, s_0)).
 \end{array}
 \tag{1}$$

**Proposition 2.2.** *Let  $R, U, S \in \text{Ob}(\Xi)$ ,  $r_0 \in \text{Legs}(R)$ ,  $u_0 \in \text{Legs}(U)$ , and  $s_0 \in \text{Legs}(S)$ . Then the diagram*

$$\begin{array}{ccc}
 \Omega(\mathcal{T}(U, u_0), \mathcal{T}(S, s_0)) \times \Omega(\mathcal{T}(R, r_0), \mathcal{T}(U, u_0)) & \xrightarrow{\circ} & \Omega(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0)) \\
 \downarrow \mathcal{A}_{u_0} \times \mathcal{A}_{s_0} & & \downarrow \mathcal{A}_{s_0} \\
 \Xi(U, S) \times \Xi(R, U) & \xrightarrow{\circ} & \Xi(R, S)
 \end{array}$$

*commutes.*

*Proof.* Let  $g \in \Omega(\mathcal{T}(U, u_0), \mathcal{T}(S, s_0))$  and  $h \in \Omega(\mathcal{T}(R, r_0), \mathcal{T}(U, u_0))$ . The diagram

$$\begin{array}{ccccc}
 & & \text{---} \iota(g \circ h) \text{---} & & \\
 & \swarrow & & \searrow & \\
 \iota(\mathcal{T}(R, r_0)) & \xrightarrow{\iota(h)} & \iota(\mathcal{T}(U, u_0)) & \xrightarrow{\iota(g)} & \iota(\mathcal{T}(S, s_0)) \\
 f_r \downarrow \cong & & f_u \downarrow \cong & & f_s \downarrow \cong \\
 R & \xrightarrow{\mathcal{A}_{u_0}(h)} & U & \xrightarrow{\mathcal{A}_{s_0}(g)} & S
 \end{array}$$

commutes, hence

$$\mathcal{A}_{s_0}(g) \circ \mathcal{A}_{u_0}(h) = f_s \circ \iota(g \circ h) \circ f_r^{-1} = \mathcal{A}_{s_0}(g \circ h).$$

□



**Lemma 2.3.** *If  $r_0 \in \text{Legs}(R)$ , then*

$$\mathcal{A}_{s_0}|_{\Omega(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0))}$$

*is injective.*

*Proof.* This is immediate from diagram (1) and the fact that the functor  $\iota$  is faithful.  $\square$

Given  $R, S \in \text{Ob}(\Xi)$ , we write  $\Xi^0(R, S) \subseteq \Xi(R, S)$  for the set of maps which factor through the vertex-free graph  $\eta$ ; if  $R$  is linear then  $\Xi^0(R, S) \cong \text{Ed}(S)$ , otherwise  $\Xi^0(R, S) = \emptyset$ .

**Definition 2.4** ('Find root' function). Let  $s_0 \in \text{Legs}(S)$ . Define a function  $\odot_{s_0}$

$$\begin{aligned} \Xi(R, S) \setminus \Xi^0(R, S) &\xrightarrow{\odot_{s_0}} \text{Legs}(R) \\ \phi &\longmapsto r_\phi, \end{aligned}$$

where  $r_\phi$  minimizes the function  $d(\phi_0(-), s_0)|_{\text{Legs}(R)}$ .

Notice that

$$(2) \quad \min_{e \in \text{Legs}(R)} d(\phi_0(e), s_0) = \min_{e \in \text{Ed}(R)} d(\phi_0(e), s_0).$$

It is clear that ' $\geq$ ' holds since  $\text{Legs}(R) \subset \text{Ed}(R)$ . On the other hand, if  $e' \in \text{Int}(R)$ , then

$$\min_{e \in \text{Legs}(R)} d(\phi_0(e), s_0) \leq d(\phi_0(e'), s_0)$$

since the minimal path from  $\phi_0(e')$  to  $s_0$  in  $\text{Legs}(S)$  must contain an element of  $\text{Legs}(\text{Im}(\phi)) = \phi_0(\text{Legs}(R))$ . Hence ' $\leq$ ' also holds in (2). In other words,  $\odot_{s_0}(\phi)$  is the edge of  $R$  which minimizes the function  $d(\phi_0(-), s_0)$ .

We now turn to several lemmas which give us effective tools for computing the function  $\odot_{s_0}$ , in particular with respect to subgraph inclusions and certain compositions.

**Lemma 2.5.** *Let  $\phi : R \rightarrow S$  be a map in  $\Xi$  which does not factor through  $\eta$ . If  $s_0 \in \text{Legs}(S)$ , then*

$$\phi_0(\odot_{s_0}(\phi)) = \odot_{s_0}(\text{Im}(\phi) \hookrightarrow S).$$

*Proof.* Write  $\tilde{e}_0 = \odot_{s_0}(\text{Im}(\phi) \hookrightarrow S) \in \text{Legs}(\text{Im}(\phi)) = \phi_0(\text{Legs}(R))$  and  $r_0 = \odot_{s_0}(\phi)$ . Then, by definition of  $\odot_{s_0}$ , for each  $e \in \text{Legs}(R)$  we have  $d(\phi_0(e), s_0) \geq d(\phi_0(r_0), s_0)$  and, for each  $\tilde{e} \in \phi_0(\text{Legs}(R))$ , we have  $d(\tilde{e}, s_0) \geq d(\tilde{e}_0, s_0)$ . It follows that  $\tilde{e}_0 = \phi_0(r_0)$ .  $\square$

In a similar vein, we have the following lemma.

**Lemma 2.6.** *Let  $\phi : R \rightarrow S \neq \eta$  be a morphism of  $\Xi$  with  $\text{Im}(\phi) = S$ . Suppose  $s_0 \in \text{Legs}(R)$  and write  $r_0 = \odot_{s_0}(\phi)$  (i.e.  $\phi_0^{-1}(s_0) \cap \text{Legs}(R) = \{r_0\}$ ). Then, for  $v \in \text{Vt}(R)$ ,*

$$\phi_0\left(\odot_{s_0}\left(\star_v \hookrightarrow R \xrightarrow{\phi} S\right)\right) = \phi_0(\odot_{r_0}(\star_v \hookrightarrow R)).$$

*In particular, if  $\phi_1(v)$  is not an edge, then*

$$\odot_{s_0}\left(\star_v \hookrightarrow R \xrightarrow{\phi} S\right) = \odot_{r_0}(\star_v \hookrightarrow R).$$

*Proof.* The second statement follows from the first since  $\phi_0|_{\text{Nbhd}(v)}$  is injective whenever  $\phi_1(v)$  is not an edge.

Suppose that the elements  $e_0 = \odot_{s_0}(\star_v \rightarrow R \rightarrow S)$  and  $e_1 = \odot_{r_0}(\star_v \rightarrow R)$  are distinct. Let  $e_1 v_2 e_2 \dots e_{n-1} v_n e_n$  be the minimal path from  $e_1$  to  $e_n = r_0 = \odot_{s_0}(\phi)$ . Then  $e_0 v_1 e_1 v_2 \dots v_n e_n$  is the minimal path from  $e_0$  to  $e_n$ , where  $v_1 = v$ . For  $i = 1, \dots, n$ , let  $P_i$  be the minimal path from  $\phi_0(e_{i-1})$  to  $\phi_0(e_i)$ ; the path  $P_i$  is a path in  $\phi_1(v_i)$ . Since the  $v_i$  are distinct,  $P_1 P_2 \dots P_n$  contains no repeated entries, hence is the minimal path from  $\phi_0(e_0)$  to  $\phi_0(r_0) = s_0$ . Likewise,  $P_2 \dots P_n$  is the minimal path from  $\phi_0(e_1)$  to  $s_0$ . Since  $d(\phi_0(e_0), s_0) \leq d(\phi_0(e_1), s_0)$ , the paths  $P_1 P_2 \dots P_n$  and  $P_2 \dots P_n$  are equal. This implies that the path  $P_1$  from  $\phi_0(e_0)$  to  $\phi_0(e_1)$  does not contain a vertex, hence  $\phi_0(e_0) = \phi_0(e_1)$ .  $\square$

**Lemma 2.7.** *If  $R$  is a subgraph of  $S$ ,  $s_0 \in \text{Legs}(S)$ , and  $r_0 = \odot_{s_0}(R \hookrightarrow S)$ , then*

$$d(r, s_0) = d(r, r_0) + d(r_0, s_0)$$

for any  $r \in \text{Ed}(R)$ .

*Proof.* Let  $P$  be the minimal path from  $r_0$  to  $s_0$ . Then  $P$  contains no edges of  $R$  other than  $r_0$  by the definition of  $\odot_{s_0}$ . Let  $P'$  be the minimal path from  $r$  to  $r_0$ . Since  $P'$  is actually a path in  $R$ , we have that  $P$  and  $P'$  only have the single edge  $r_0$  in common. Thus  $P'P$  has no repeated edges, and thus is a minimal path from  $r$  to  $s_0$  by the characterization in proposition 1.17. Thus

$$\begin{aligned} d(r, s_0) &= \frac{|P'P| - 1}{2} = \frac{(|P'| + |P| - 1) - 1}{2} \\ &= \frac{|P'| - 1}{2} + \frac{|P| - 1}{2} = d(r, r_0) + d(r_0, s_0). \end{aligned}$$

$\square$

**Lemma 2.8.** *Suppose that*

$$A \hookrightarrow B \hookrightarrow C$$

*is a pair of subgraph inclusions,  $c_0 \in \text{Legs}(C)$ , and  $b_0 = \odot_{c_0}(B \hookrightarrow C) \in \text{Legs}(B)$ . Then  $\odot_{b_0}(A \hookrightarrow B) = \odot_{c_0}(A \hookrightarrow C)$ .*

*Proof.* Write  $a_1 = \odot_{b_0}(A \hookrightarrow B)$  and apply lemma 2.7 twice to get

$$(3) \quad d(a, b_0) = d(a, a_1) + d(a_1, b_0)$$

$$(4) \quad d(a, c_0) = d(a, b_0) + d(b_0, c_0)$$

for any  $a \in \text{Ed}(A) \subset \text{Ed}(B)$ . For the particular case when  $a = a_1$ , (4) becomes  $d(a_1, c_0) = d(a_1, b_0) + d(b_0, c_0)$ . Combining with (3) we have

$$d(a, b_0) = d(a, a_1) + d(a_1, c_0) - d(b_0, c_0),$$

hence

$$d(a, c_0) \stackrel{(4)}{=} d(a, b_0) + d(b_0, c_0) = d(a, a_1) + d(a_1, c_0).$$

Then  $a_1$  is the element of  $\text{Ed}(A)$  which minimizes this function, hence

$$a_1 = \odot_{c_0}(A \hookrightarrow C).$$

$\square$

### 2.1. Orientation of maps.

**Lemma 2.9.** *Let  $r_0 \in \text{Ed}(R)$  and  $s_0 \in \text{Ed}(S)$ . There is a function  $\mathcal{L} = \mathcal{L}_{s_0, r_0}$  so that the following diagram commutes.*

$$\begin{array}{ccc} \odot_{s_0}^{-1}(r_0) & \xrightarrow{\mathcal{L}} & \Omega(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0)) \\ \downarrow & & \downarrow \mathcal{A}_{s_0} \\ \Xi(R, S) \setminus \Xi^0(R, S) & \xrightarrow{\quad} & \Xi(R, S) \end{array}$$

*Proof.* Let  $\phi : R \rightarrow S$  be in  $\odot_{s_0}^{-1}(r_0)$ . To see that the functions

$$\begin{aligned} \phi_0 &: \text{Ed}(R) \rightarrow \text{Ed}(S) \\ \phi_1 &: \text{Vt}(R) \rightarrow \text{Sbgph}(S) \end{aligned}$$

determine a map  $\mathcal{T}(R, r_0) \rightarrow \mathcal{T}(S, s_0)$  in  $\Omega$ , we just need to establish that, for each  $v \in \text{Vt}(R)$  we have  $\phi_0(\text{out}(v)) = \text{out}(\phi_1(v))$ . There is nothing to prove in the case  $\phi_1(v)$  is a single edge. For concision, write  $\bar{r}_0$  for the element

$$\bar{r}_0 := \phi_0(r_0) = \phi_0(\odot_{s_0}(\phi)) \stackrel{2.5}{=} \odot_{s_0}(\text{Im}(\phi) \hookrightarrow S) \in \text{Legs}(\text{Im}(\phi)) = \phi_0(\text{Legs}(R)).$$

In the case when  $\phi_1(v)$  is not an edge, we have

$$\begin{aligned} \text{out}(\phi_1(v)) &= \odot_{s_0}(\phi_1(v) \hookrightarrow S) \\ &= \odot_{\bar{r}_0}(\phi_1(v) \hookrightarrow \text{Im}(\phi)) && \text{Lemma 2.8} \\ &= \phi_0(\odot_{\bar{r}_0}(\star_v \hookrightarrow R \rightarrow \text{Im}(\phi))) && \text{Lemma 2.5} \\ &= \phi_0(\odot_{r_0}(\star_v \hookrightarrow R)) && \text{Lemma 2.6} \\ &= \phi_0(\text{out}(v)). \end{aligned}$$

□

**Lemma 2.10.** *If  $r_0 \neq r_1 \in \text{Ed}(R)$ , then*

$$\left( \begin{array}{c} \mathcal{A}_{s_0} \left( \Omega(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0)) \right) \\ \cap \mathcal{A}_{s_0} \left( \Omega(\mathcal{T}(R, r_1), \mathcal{T}(S, s_0)) \right) \end{array} \right) \setminus \Xi^0(R, S) = \emptyset.$$

*Proof.* If  $\phi \in \mathcal{A}_{s_0} \left( \Omega(\mathcal{T}(R, r_i), \mathcal{T}(S, s_0)) \right) \setminus \Xi^0(R, S)$ , then  $\odot_{s_0}(\phi) = r_i$ . □

The following proposition describes precisely how the functions  $\mathcal{L}_{s_0, r_0}$  behave with respect to composition in  $\Xi$ . Special cases of the first part have already appeared in lemmas 2.6 and 2.8.

**Proposition 2.11.** *Consider a composite*

$$R \xrightarrow{\psi} U \xrightarrow{\phi} S$$

*in  $\Xi$ , and let  $s_0 \in \text{Legs}(S)$ . If  $u_0 = \odot_{s_0}(\phi)$ , then*

$$\odot_{u_0}(\psi) = \odot_{s_0}(\phi \circ \psi)$$

*and*

$$\mathcal{L}_{s_0}(\phi) \circ \mathcal{L}_{u_0}(\psi) = \mathcal{L}_{s_0}(\phi \circ \psi).$$

*Proof.* Set  $u_0 = \odot_{s_0}(\phi)$  and  $r_0 = \odot_{u_0}(\psi)$ . By proposition 2.2 and lemma 2.9, we know that the diagram

$$\begin{array}{ccc}
 & \odot_{s_0}^{-1}(u_0) \times \odot_{u_0}^{-1}(r_0) & \\
 \swarrow \mathcal{L}_{u_0} \times \mathcal{L}_{s_0} & & \searrow \\
 \Omega(\mathcal{T}(U, u_0), \mathcal{T}(S, s_0)) \times \Omega(\mathcal{T}(R, r_0), \mathcal{T}(U, u_0)) & \xrightarrow{\mathcal{A}_{u_0} \times \mathcal{A}_{s_0}} & \Xi(U, S) \times \Xi(R, U) \\
 \downarrow \circ & & \downarrow \circ \\
 \Omega(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0)) & \xrightarrow{\mathcal{A}_{s_0}} & \Xi(R, S)
 \end{array}$$

commutes. Further,  $(\phi, \psi)$  is an element in the apex. Thus

$$\phi \circ \psi \in \mathcal{A}_{s_0} \left( \Omega(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0)) \right),$$

so  $\odot_{s_0}(\phi \circ \psi) = r_0$ . Thus we have established the first statement.

The second statement is now immediate since the underlying maps involved are just  $\phi_i$ ,  $\psi_i$ , and  $(\phi \circ \psi)_i$  by the proof of lemma 2.9.  $\square$

**Theorem 2.12.** *Suppose that  $s_0 \in \text{Legs}(S)$ . The function*

$$\mathcal{A}_{s_0} : \coprod_{\text{Legs}(R)} \Omega(\mathcal{T}(R, r), \mathcal{T}(S, s_0)) \rightarrow \Xi(R, S)$$

*restricts to a bijection*

$$\begin{aligned}
 \mathcal{A}_{s_0}^{nc} : \coprod_{\text{Legs}(R)} \left( \Omega(\mathcal{T}(R, r), \mathcal{T}(S, s_0)) \setminus \Omega^0(\mathcal{T}(R, r), \mathcal{T}(S, s_0)) \right) &\rightarrow \Xi(R, S) \setminus \Xi^0(R, S) \\
 &= \coprod_{\text{Legs}(R)} \odot_{s_0}^{-1}(r).
 \end{aligned}$$

*Proof.* There is a diagram

$$\begin{array}{ccc}
 \coprod_{\text{Legs}(R)} \left( \Omega(\mathcal{T}(R, r), \mathcal{T}(S, s_0)) \setminus \Omega^0(\mathcal{T}(R, r), \mathcal{T}(S, s_0)) \right) & \hookrightarrow & \coprod_{\text{Legs}(R)} \Omega(\mathcal{T}(R, r), \mathcal{T}(S, s_0)) \\
 \downarrow \mathcal{A}_{s_0}^{nc} & \nearrow \mathcal{L}_{s_0} & \downarrow \mathcal{A}_{s_0} \\
 \Xi(R, S) \setminus \Xi^0(R, S) & \hookrightarrow & \Xi(R, S).
 \end{array}$$

The bottom triangle commutes by lemma 2.9. Given  $\psi : R \rightarrow S$ ,  $\mathcal{L}_{s_0}(\psi) : \mathcal{T}(R, r_0) \rightarrow \mathcal{T}(S, s_0)$  is the unique map of  $\Omega$  so that  $\mathcal{A}_{s_0}(\mathcal{L}_{s_0}(\psi)) = \psi$ . If  $\psi = \mathcal{A}_{s_0}(\phi : \mathcal{T}(R, r_0) \rightarrow \mathcal{T}(S, s_0))$ , then certainly  $\phi$  satisfies the condition to be  $\mathcal{L}_{s_0}(\mathcal{A}_{s_0}(\phi))$ . Thus the top triangle commutes.

Since  $\mathcal{L}_{s_0}$  is injective, so is  $\mathcal{A}_{s_0}^{nc}$ . Further, if  $\phi : R \rightarrow S$  is not constant, then

$$\mathcal{L}_{s_0}(\phi) \in \Omega(\mathcal{T}(R, \odot_{s_0}(\phi)), \mathcal{T}(S, s_0))$$

is also not constant, so  $\mathcal{A}_{s_0}^{nc}$  is surjective.  $\square$

**Corollary 2.13.** *We have, for each  $s_0 \in \text{Legs}(S)$*

$$\text{Iso}_{\Xi}(R, S) \cong \coprod_{\text{Legs}(R)} \text{Iso}_{\Omega}(\mathcal{T}(R, r), \mathcal{T}(S, s_0))$$

via  $\mathcal{A}_{s_0}$ . Specializing to the case  $R = S$ , we have

$$\text{Aut}_{\Xi}(S) \cong \coprod_{s \in \text{Legs}(S)} \text{Iso}_{\Omega}(\mathcal{T}(S, s), \mathcal{T}(S, s_0)).$$

*Proof.* The statement is trivial if  $S$  (and hence  $R$ ) does not have a vertex. Otherwise, isomorphisms are not constant, so this follows from theorem 2.12 by taking subsets.  $\square$

**Example 2.14.** Consider the tree  $S$  with  $\text{Ed}(S) = \text{Legs}(S) = \{e_1, e_2, e_3\}$  and  $\text{Vt}(S) = \{v\}$ . We can apply corollary 2.13 to reveal some structure of  $\text{Aut}_{\Xi}(S) = \Sigma_3$ , but  $\mathcal{A}_{e_i}$  as  $i$  varies do not behave well together. For example, under the composite

$$\begin{array}{c} \text{Iso}_{\Omega}(\mathcal{T}(S, e_1), \mathcal{T}(S, e_2)) \xlongequal{\quad} \{\phi, \psi\} \\ \downarrow \\ \coprod_{i=1}^3 \text{Iso}_{\Omega}(\mathcal{T}(S, e_i), \mathcal{T}(S, e_2)) \xrightarrow[\cong]{\mathcal{A}_{e_2}} \text{Aut}_{\Xi}(S) \xrightarrow[\cong]{\mathcal{A}_{e_1}^{-1}} \coprod_{j=1}^3 \text{Iso}_{\Omega}(\mathcal{T}(S, e_j), \mathcal{T}(S, e_1)) \end{array}$$

(where, say,  $\phi_0(e_1) = e_2, \phi_0(e_2) = e_1, \phi_0(e_3) = e_3, \psi_0(e_1) = e_2, \psi_0(e_2) = e_3, \psi_0(e_3) = e_1$ )  $\phi$  and  $\psi$  map to different coproduct summands. The morphism  $\phi$  lands in the  $j = 2$  component and  $\psi$  lands in the  $j = 3$  component.

**Corollary 2.15.** *If  $R$  is non-linear, then*

$$\mathcal{A}_{s_0} : \coprod_{\text{Legs}(R)} \Omega(\mathcal{T}(R, r), \mathcal{T}(S, s_0)) \rightarrow \Xi(R, S)$$

*is an isomorphism, with inverse given by  $\mathcal{L}_{s_0}$ .*

*Proof.* If  $R$  is non-linear, then  $\Xi^0(R, S) = \emptyset$ .  $\square$

**Proposition 2.16.** *If  $R \neq \eta$  is a linear graph with at least one vertex, then  $\text{Legs}(R) = \{r_0, r_1\}$  and*

$$\mathcal{A}_{s_0} : \Omega(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0)) \amalg \Omega(\mathcal{T}(R, r_1), \mathcal{T}(S, s_0)) \rightarrow \Xi(R, S)$$

*satisfies*

$$|\mathcal{A}_{s_0}^{-1}(\phi)| = \begin{cases} 2 & \phi \in \Xi^0(R, S) \\ 1 & \phi \notin \Xi^0(R, S). \end{cases}$$

*If  $R = \eta = L_0$ , then  $\mathcal{A}_{s_0}$  is a bijection.*

*Proof.* In general,  $\mathcal{A}_{s_0}$  splits as a coproduct of  $\mathcal{A}_{s_0}^{nc}$  with

$$\mathcal{A}_{s_0}^c : \coprod_{\text{Legs}(R)} \Omega^0(\mathcal{T}(R, r), \mathcal{T}(S, s_0)) \rightarrow \Xi^0(R, S).$$

If  $R$  is linear and has at least one vertex, then

$$\begin{array}{ccc} \Omega^0(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0)) \amalg \Omega^0(\mathcal{T}(R, r_1), \mathcal{T}(S, s_0)) & \xrightarrow{\mathcal{A}_{s_0}^c} & \Xi^0(R, S) \\ \downarrow \cong & & \downarrow \cong \\ \text{Ed}(S) \amalg \text{Ed}(S) & \xrightarrow{\text{id} \amalg \text{id}} & \text{Ed}(S) \end{array}$$

is two-to-one and  $\mathcal{A}_{s_0}^{nc}$  is injective by theorem 2.12. Hence the first statement is proved.

For the second statement, if  $R = \eta$  then all maps  $R \rightarrow S$  are constant,  $\text{Legs}(R) = \{r\}$  has one element, and

$$\mathcal{A}_{s_0} = \mathcal{A}_{s_0}^c : \Omega^0(\mathcal{T}(R, r), \mathcal{T}(S, s_0)) \rightarrow \Xi^0(R, S)$$

is isomorphic to the identity map on  $\text{Ed}(S)$ . □

### 3. A GENERALIZED REEDY STRUCTURE ON $\Xi$

The category  $\Delta$  of nonempty finite ordered sets is the prototypical example of a *Reedy category*. The surjective (resp. injective) maps form a wide subcategory<sup>3</sup>  $\Delta^-$  (resp.  $\Delta^+$ ) of morphisms which lower (resp. raise) degrees, such that any map has a unique factorization  $f = f^+ f^-$ . Numerous inductive techniques in used in the theory of (co)simplicial objects actually work in diagrams indexed by arbitrary Reedy categories.

Generalized Reedy categories were introduced in [3] and capture the dendroidal category  $\Omega$  as an example, highlighting its similarities to  $\Delta$ . We will return to the theory of model structures on diagram categories index by a generalized Reedy category  $\mathbb{R}$  in section 8, but for now we show that  $\Xi$  admits such a structure.

**Definition 3.1.** A *generalized Reedy structure* on a small category  $\mathbb{R}$  consists of

- wide subcategories  $\mathbb{R}^+$  and  $\mathbb{R}^-$ , and
- a degree function  $d : \text{Ob}(\mathbb{R}) \rightarrow \mathbb{N}$

satisfying the following four axioms.

- (i) Non-invertible morphisms in  $\mathbb{R}^+$  (resp.,  $\mathbb{R}^-$ ) raise (resp., lower) the degree. Isomorphisms in  $\mathbb{R}$  preserve the degree.
- (ii)  $\mathbb{R}^+ \cap \mathbb{R}^- = \text{Iso}(\mathbb{R})$ .
- (iii) Every morphism  $f$  of  $\mathbb{R}$  factors as  $f = gh$  with  $g \in \mathbb{R}^+$  and  $h \in \mathbb{R}^-$ , and this factorization is unique up to isomorphism.
- (iv) If  $\theta f = f$  for  $\theta \in \text{Iso}(\mathbb{R})$  and  $f \in \mathbb{R}^-$ , then  $\theta$  is an identity.

If, moreover, the condition

- (iv') If  $f\theta = f$  for  $\theta \in \text{Iso}(\mathbb{R})$  and  $f \in \mathbb{R}^+$ , then  $\theta$  is an identity

holds, then we call this a generalized *dualizable* Reedy structure.

An ordinary Reedy category is a generalized Reedy category where there are no isomorphisms other than the identity maps.

**Definition 3.2.** Consider the following structures on  $\Xi$ .

- A degree function

$$\begin{aligned} d : \text{Ob}(\Xi) &\rightarrow \mathbb{N} \\ S &\mapsto |\text{Vt}(S)| \end{aligned}$$

- a wide subcategory  $\Xi^+$  consisting of all maps  $\phi : R \rightarrow S$  so that  $\phi_0 : \text{Ed}(R) \rightarrow \text{Ed}(S)$  is injective.

---

<sup>3</sup>A wide subcategory is a subcategory which contains every object of the ambient category.

- a wide subcategory  $\Xi^-$  consisting of all maps  $\phi: R \rightarrow S$  so that  $\phi_0$  is surjective and, for each vertex  $v \in \text{Vt}(S)$ , there exists a vertex  $w \in \text{Vt}(R)$  with  $v \in \text{Vt}(\phi_1(w))$ .

This definition is chosen to be compatible with the known generalized Reedy structure<sup>4</sup> on the dendroidal category  $\Omega$  from definition 1.19, in the sense that the following equalities hold

$$\begin{aligned}\Omega^+ &= \Omega \cap \Xi^+ \\ \Omega^- &= \Omega \cap \Xi^- \quad (d : \text{Ob}(\Omega) \rightarrow \mathbb{N}) = (d : \text{Ob}(\Xi) \rightarrow \mathbb{N})|_{\text{Ob}(\Omega)}.\end{aligned}$$

Notice that

$$(5) \quad \Xi^0(R, S) \cap \Xi^+(R, S) \cong \begin{cases} \text{Ed}(S) & \text{if } R = \eta \\ \emptyset & \text{otherwise} \end{cases}$$

is nonempty if and only if  $R = \eta$ .

**Lemma 3.3.** *Given  $s_0 \in \text{Ed}(S)$ , the map*

$$\mathcal{L} : \Xi(R, S) \setminus \Xi^0(R, S) \rightarrow \coprod_{r \in \text{Ed}(R)} \Omega(\mathcal{T}(R, r), \mathcal{T}(S, s_0))$$

*restricts to maps*

$$\begin{aligned}\Xi^+(R, S) \setminus \Xi^0(R, S) &\rightarrow \coprod_{r \in \text{Ed}(R)} \Omega^+(\mathcal{T}(R, r), \mathcal{T}(S, s_0)) \\ \Xi^-(R, S) \setminus \Xi^0(R, S) &\rightarrow \coprod_{r \in \text{Ed}(R)} \Omega^-(\mathcal{T}(R, r), \mathcal{T}(S, s_0))\end{aligned}$$

**Proposition 3.4.** *With the structure from definition 3.2,  $\Xi$  is a dualizable generalized Reedy category.*

*Proof.* For (i): note that isomorphisms preserve degree. If  $\phi: R \rightarrow S \in \Xi^+$ , then  $\{\text{Vt}(\phi_1(v))\}_{v \in \text{Vt}(R)}$  is a collection of pairwise disjoint, non-empty subsets of  $\text{Vt}(S)$ . Thus

$$d(R) = |\text{Vt}(R)| = \sum_{v \in \text{Vt}(R)} 1 \leq \sum_{v \in \text{Vt}(R)} |\text{Vt}(\phi_1(v))| \leq |\text{Vt}(S)| = d(S).$$

If  $\phi: R \rightarrow S \in \Xi^-$ , then surjectivity of  $\phi_0$  implies (by lemma 1.22) that  $|\text{Vt}(\phi_1(v))| \leq 1$  for all  $v \in \text{Vt}(R)$ . Let  $\text{Vt}(S) \rightarrow \text{Vt}(R)$  be the map which sends  $v \in \text{Vt}(S)$  to the (unique, by definition 1.12(3)) vertex  $w$  with  $v \in \text{Vt}(\phi_1(w))$ . Since there is at most one  $v$  in a given  $\phi_1(w)$ , the map  $\text{Vt}(S) \rightarrow \text{Vt}(R)$  is injective, hence  $d(S) \leq d(R)$ .

For (ii), it is clear that  $\text{Iso}(\Xi)$  is contained in  $\Xi^+ \cap \Xi^-$ . For the reverse inclusion, suppose that  $\phi: R \rightarrow S$  is in both  $\Xi^+$  and  $\Xi^-$ . If  $\phi$  is constant, then by (5) we must have  $R = \eta$ ; since  $\phi$  is also in  $\Xi^-$ ,  $S$  is also an edge, and  $\phi$  is an isomorphism. If  $\phi$  is not constant, choose a root  $s_0$  for  $S$ . Then by lemma 3.3 we know that

$$\mathcal{L}(\phi) \in \Omega^+(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0)) \cap \Omega^-(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0)) = \text{Iso}_\Omega(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0))$$

where  $r_0 = \odot_{s_0}(\phi)$ . By corollary 2.13, we thus have that  $\phi$  is an isomorphism in  $\Xi$ . Since  $\phi$  was arbitrary, we have  $\Xi^+ \cap \Xi^- \subseteq \text{Iso}(\Xi)$  as well.

For (iv), note that if  $\theta$  is an isomorphism,  $\phi: R \rightarrow S \in \Xi^-$ , and  $\theta\phi = \phi$ , then  $\theta_0\phi_0 = \phi_0$ . Since  $\theta_0$  is a bijection of sets and  $\phi_0$  is a surjection of sets, it follows

<sup>4</sup>See [3, Example 2.8] and the minor correction in [10, p. 216].

that  $\theta_0$  is an identity. There is only one isomorphism  $S \rightarrow S$  in  $\Xi$  which is the identity on edges (example 1.13), hence  $\theta = \text{id}_S$ . The proof that  $\Xi$  satisfies (iv') follows similarly.

We finally turn to (iii). We first *construct* a factorization a given morphism of  $\Xi$ . We may assume that  $\phi \in \Xi(R, S) \setminus \Xi^0(R, S)$ . Pick a root  $s_0$  for  $S$ , and consider

$$\mathcal{L}(\phi) \in \Omega(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0))$$

where  $r_0 = \odot_{s_0}(\phi)$ . Then there is a decomposition

$$\mathcal{L}(\phi) = g \circ h$$

with

$$g \in \Omega^+(T, \mathcal{T}(S, s_0)) \quad h \in \Omega^-(\mathcal{T}(R, r_0), T).$$

We have

$$\begin{array}{ccccc} & & \mathcal{L}(\phi) & & \\ & \nearrow & & \searrow & \\ \mathcal{T}(R, r_0) & \xrightarrow{h} & T & \xrightarrow{g} & \mathcal{T}(S, s_0); \end{array}$$

apply the functor  $\iota$  to this diagram to get

$$\begin{array}{ccccc} & & \iota(\mathcal{L}(\phi)) & & \\ & \nearrow & & \searrow & \\ \iota(\mathcal{T}(R, r_0)) & \xrightarrow{\iota h} & \iota T & \xrightarrow{\iota g} & \iota(\mathcal{T}(S, s_0)) \\ \downarrow \cong & & & & \downarrow \cong \\ R & \xrightarrow{\mathcal{A}(\mathcal{L}(\phi)) = \phi} & & & S. \end{array}$$

Since  $\iota h \in \Xi^-$ ,  $\iota g \in \Xi^+$ , and isomorphisms are in  $\Xi^+ \cap \Xi^-$ , we have provided the desired decomposition of  $\phi$ .

Suppose that  $\phi^1 \circ \psi^1 = \phi^2 \circ \psi^2$  with  $\phi^i \in \Xi^+(U_i, S)$  and  $\psi^i \in \Xi^-(R, U_i)$ . Let  $u_i = \odot_{s_0}(\psi^i)$  and  $r_0 = \odot_{u_1}(\phi^1) \stackrel{2.11}{=} \odot_{u_2}(\phi^2)$ . We have, by proposition 2.11,

$$\mathcal{L}_{u_1}(\phi^1) \circ \mathcal{L}_{s_0}(\psi^1) = \mathcal{L}_{u_2}(\phi^2) \circ \mathcal{L}_{s_0}(\psi^2).$$

Now  $\mathcal{L}_{u_i}(\phi^i) \in \Omega^+$  and  $\mathcal{L}_{s_0}(\psi^i) \in \Omega^-$ , so there exist isomorphisms  $a_i$  making the diagram

$$\begin{array}{ccccc} \mathcal{T}(R, r_0) & \xrightarrow{\mathcal{L}_{u_1}(\psi^1)} & \mathcal{T}(U_1, u_1) & \xrightarrow{\mathcal{L}_{s_0}(\phi^1)} & \mathcal{T}(S, S_0) \\ \cong \downarrow a_1 & & \cong \downarrow a_2 & & \cong \downarrow a_3 \\ \mathcal{T}(R, r_0) & \xrightarrow{\mathcal{L}_{u_2}(\psi^2)} & \mathcal{T}(U_2, u_2) & \xrightarrow{\mathcal{L}_{s_0}(\phi^2)} & \mathcal{T}(S, S_0) \end{array}$$

commute. Applying  $\iota$  gives the back square of the diagram;

$$\begin{array}{ccccccc} \iota(\mathcal{T}(R, r_0)) & \xrightarrow{\iota(\mathcal{L}_{u_1}(\psi^1))} & \iota(\mathcal{T}(U_1, u_1)) & \xrightarrow{\iota(\mathcal{L}_{s_0}(\phi^1))} & \iota(\mathcal{T}(S, S_0)) & & \\ \cong \downarrow \iota(a_1) & \nearrow & \downarrow \cong & \nearrow & \downarrow \cong & \nearrow & \\ R & \xrightarrow{\psi^1} & U_1 & \xrightarrow{\phi^1} & S & & \\ \downarrow \cong & & \downarrow \cong \iota(a_2) & & \downarrow \cong \iota(a_3) & & \\ \iota(\mathcal{T}(R, r_0)) & \xrightarrow{\iota(\mathcal{L}_{u_2}(\psi^2))} & \iota(\mathcal{T}(U_2, u_2)) & \xrightarrow{\iota(\mathcal{L}_{s_0}(\phi^2))} & \iota(\mathcal{T}(S, S_0)) & & \\ \downarrow \cong & \nearrow & \downarrow \cong & \nearrow & \downarrow \cong & \nearrow & \\ R & \xrightarrow{\psi^2} & U_2 & \xrightarrow{\phi^2} & S & & \end{array}$$



there exist dashed maps making the diagram commute. Each of these maps is necessarily an isomorphism. Thus, the front face establishes uniqueness of decompositions in  $\Xi$ .  $\square$

#### 4. THE ACTIVE / INERT WEAK FACTORIZATION SYSTEM ON $\Xi$

In this section we exhibit a weak factorization system on the category  $\Xi$ .

Given a class  $I$  of morphisms in a category  $\mathcal{C}$ , write  $I^\square$  for the maps which have the right lifting property with respect to every element of  $I$ . In other words,  $f : X \rightarrow Y$  is in  $I^\square$  if and only if every commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow i & \nearrow & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

with  $i \in I$  admits a lift  $B \rightarrow X$ . Similarly,  ${}^\square I$  is the class of maps having the left-lifting property with respect to every element of  $I$ .

A weak factorization system (see [36, 11.2.1]) consists of two classes of maps  $\mathcal{L}$  and  $\mathcal{R}$  so that every morphism factors into a map in  $\mathcal{R}$  followed by one in  $\mathcal{L}$ , and so that

$${}^\square \mathcal{R} = \mathcal{L} \quad \& \quad \mathcal{L}^\square = \mathcal{R}.$$

**Definition 4.1.** A morphism  $\phi : R \rightarrow S$  in  $\Xi$  is called

- *active* if  $\text{Legs}(S) \subseteq \text{Im}(\phi_0)$ ; and
- *inert* if  $\phi_0 : \text{Ed}(R) \rightarrow \text{Ed}(S)$  is injective and if  $w \in \text{Vt}(\phi_1(v))$  then  $\text{Nbhd}(w) \in \text{Im}(\phi_0)$ .

Notice that every map in  $\Xi^-$  (see definition 3.2) is an active map. Further, every inert map is contained in  $\Xi^+$  by the following remark.

**Remark 4.2.** Suppose that  $\phi$  is inert. Then, since  $\phi_0$  is injective, we know  $0 < |\text{Vt}(\phi_1(v))|$  for any  $v \in \text{Vt}(R)$ . Further, if  $\text{Vt}(\phi_1(v))$  contains two (adjacent) vertices  $w_1$  and  $w_2$ , connected by an edge  $e$ , then  $e \in \text{Int}(\phi_1(v)) \cap \text{Nbhd}(w_1) \subset \text{Int}(\phi_1(v)) \cap \text{Im}(\phi_0) = \emptyset$  by lemma 1.23. Thus  $|\text{Vt}(\phi_1(v))| \leq 1$ . In other words,  $\phi$  is (isomorphic to) a subgraph inclusion.

**Lemma 4.3.** *The set of active morphisms has the left lifting property with respect to the set of inert morphisms.*

*Proof.* Suppose we are given a commutative diagram in  $\Xi$

$$(6) \quad \begin{array}{ccc} R & \xrightarrow{\alpha} & P \\ \downarrow \phi & & \downarrow \psi \\ S & \xrightarrow{\beta} & Q \end{array}$$

with  $\phi$  active and  $\psi$  inert (so, in particular,  $P$  is a subgraph of  $Q$ ).

If  $S = \eta$  consists of the single edge  $e$ , then  $R$  is linear. We have a diagram

$$\begin{array}{ccc} \text{Legs}(R) & \xrightarrow{\alpha_0|_{\text{Legs}(R)}} & \text{Ed}(P) \\ \downarrow \phi_0 & & \downarrow \psi_0 \\ \{e\} & \xrightarrow{\beta_0} & \text{Ed}(Q) \end{array}$$

with  $\psi_0$  an injection. Thus  $\text{Im}(\alpha_0|_{\text{Legs}(R)}) = |_{e'}$  is a single edge, and we define  $\gamma : S \rightarrow P$  by  $\gamma_0(e) = e'$ .

Suppose now that  $S$  contains a vertex; then we have that

$$\phi_0|_{\text{Legs}(R)} : \text{Legs}(R) \rightarrow \text{Legs}(S)$$

is a bijection. Define  $\gamma_0|_{\text{Legs}(S)} = \alpha_0 \circ (\phi_0|_{\text{Legs}(R)})^{-1}$ . We wish to extend this to  $\text{Int}(S)$ , which we will do in a moment. Since  $\phi$  is active, every  $w \in \text{Vt}(S)$  is in  $\text{Vt}(\phi_1(v))$  for some  $v \in \text{Vt}(R)$ . Notice that

$$\bigcup_{w \in \phi_1(v)} \beta_1(w) = (\beta \circ \phi)_1(v) = (\psi \circ \alpha)_1(v) = \bigcup_{t \in \alpha_1(v)} \psi_1(t) \in \text{Sbgph}(\text{Im}(\psi)) \xleftarrow{\cong} \text{Sbgph}(P)$$

since  $P \rightarrow \text{Im}(\psi)$  is an isomorphism. Thus, for  $w \in \text{Vt}(\phi_1(v))$ , there is a subgraph  $G_w \in \text{Sbgph}(P)$  so which maps to  $\beta_1(w) \in \text{Sbgph}(\text{Im}(\psi)) \subset \text{Sbgph}(Q)$  under  $\psi$ .

Suppose that  $e \in \text{Int}(S)$  is any edge. Since  $S$  contains a vertex,  $e$  is adjacent to two distinct vertices  $w$  and  $w'$ . We have  $\beta_1(w) \cap \beta_1(w') = \{\beta_0(e)\}$ , hence  $G_w \cap G_{w'} = \{\tilde{e}\}$  for some edge  $\tilde{e}$ . Set  $\gamma_0(e) = \tilde{e}$ ; by definition we have  $\phi_0\gamma_0(e) = \beta_0(e)$ . Further, since  $\phi$  is active it sends legs to legs, so if  $e = \phi_0(e')$ , then  $e'$  is an internal edge between distinct vertices  $v$  and  $v'$ , with  $w \in \text{Vt}(\phi_1(v))$  and  $w' \in \text{Vt}(\phi_1(v'))$ . We have  $\{\gamma_0(e)\} = G_w \cap G_{w'} \subset \alpha_1(v) \cap \alpha_1(v') = \{\alpha_0(e')\}$ , hence  $\gamma_0\phi_0(e') = \gamma_0(e) = \alpha_0(e')$ . In conclusion,  $\alpha_0 = \gamma_0\phi_0$  and  $\psi_0\gamma_0 = \beta_0$ .

Next, define

$$\gamma_1(w) = G_w \in \text{Sbgph}(\alpha_1(v)).$$

By definition, we have

$$(\psi \circ \gamma)_1(w) = \bigcup_{v \in \gamma_1(w)} \psi_1(v) = \beta_1(w),$$

hence  $\psi \circ \gamma = \beta$ . Then  $\psi \circ \gamma \circ \phi = \beta \circ \phi = \psi \circ \alpha$ , hence the injective map  $\psi : \text{Sbgph}(P) \hookrightarrow \text{Sbgph}(Q)$  takes  $(\gamma \circ \phi)_1(v)$  and  $\alpha_1(v)$  to the same element. It follows that  $(\gamma \circ \phi)_1(v) = \alpha_1(v)$ , so  $\gamma \circ \phi = \alpha$ .

Thus we have shown that (6) always has a lift.  $\square$

**Proposition 4.4.** *The active and inert morphisms form a weak factorization system.*

*Proof.* Given a map  $\phi : R \rightarrow S$ , we can factor  $\phi$  as  $R \rightarrow \text{Im}(\phi) \rightarrow S$ . The map  $R \rightarrow \text{Im}(\phi)$  is active since  $\text{Legs}(\text{Im}(\phi)) = \phi_0(\text{Legs}(R))$ , while  $\text{Im}(\phi) \rightarrow S$  is a subgraph inclusion, hence inert.

For the remainder of the proof, write  $\mathcal{L}$  for the set of active morphisms and  $\mathcal{R}$  for the set of inert morphisms. In lemma 4.3 we showed that  $\mathcal{R} \subseteq \mathcal{L}^\boxtimes$  (or, equivalently,  $\mathcal{L} \subseteq {}^\boxtimes\mathcal{R}$ ).

For the reverse inclusions, suppose that  $\phi : R \rightarrow S$  is in  $\mathcal{L}^\boxtimes$ . Consider the decomposition

$$R \xrightarrow{\phi^-} T \xrightarrow{\phi^+} S$$

coming from the generalized Reedy structure on  $\Xi$ . We know that  $\phi^-$  is an active morphism, hence we can lift in the diagram

$$\begin{array}{ccc} R & \xrightarrow{\text{id}} & R \\ \downarrow \phi^- & & \downarrow \phi \\ T & \xrightarrow{\phi^+} & S \end{array}$$

which implies  $\psi\phi^- = \text{id}$ , hence  $\phi^-$  is an isomorphism. Thus  $\phi_0$  is injective. Further, the diagram

$$\begin{array}{ccc} R & \xrightarrow{\text{id}} & R \\ \downarrow \bar{\phi} & & \downarrow \phi \\ \text{Im}(\phi) & \xrightarrow{i} & S \end{array}$$

admits a lift; if  $w$  is a vertex in  $\phi_1(v) \subset \text{Im}(\phi)$ , then

$$\text{Nbhd}(w) = i_0(\text{Nbhd}(w)) = \phi_0\psi_0(\text{Nbhd}(w)) \subset \text{Im}(\phi_0).$$

Hence  $\phi$  is inert, and we see that  $\mathcal{L}^\varnothing \subset \mathcal{R}$ .

Now suppose that  $\phi : R \rightarrow S$  is in  ${}^\varnothing\mathcal{R}$ . Then

$$\begin{array}{ccc} R & \xrightarrow{\bar{\phi}} & \text{Im}(\phi) \\ \downarrow \phi & & \downarrow i \\ S & \xrightarrow{\text{id}} & S \end{array}$$

admits a lift  $\psi : S \rightarrow \text{Im}(\phi)$ . Then

$$\text{Legs}(S) \xrightarrow{\psi_0} \text{Ed}(\text{Im}(\phi)) = \text{Im}(\phi_0) \subset \text{Ed}(S)$$

is just the usual inclusion, hence  $\text{Legs}(S) \subset \text{Im}(\phi_0)$ , so  $\phi$  is active. Since  $\phi$  was arbitrary, we know that  ${}^\varnothing\mathcal{R} \subset \mathcal{L}$ . □

The active and inert maps which are oriented form a weak factorization system on the category  $\Omega$ . This was proved in [27, 1.3.13], where inert maps are called ‘free’ and active maps are called ‘boundary preserving’.

**Proposition 4.5.** *The inclusion  $\iota : \Omega \rightarrow \Xi$  respects the weak factorization structures.* □

## 5. CYCLIC OPERADS

We now turn to a definition of colored cyclic operad [25]. We begin by considering certain collections of trees, and give a definition in section 5.2 as a category of algebras over a certain monad. Certainly other definitions are possible: using colored operads or algebraic theories, or by giving a straight axiomatic description. We furthermore show that the functor  $U : \mathbf{Cyc} \rightarrow \mathbf{Op}$  which forgets the cyclic structure is faithful, and give an example of two non-isomorphic cyclic operads that have the same underlying ordinary operad.

We should note that our definition is not a symmetric version of the ‘cyclic multicategories’ of Cheng, Gurski, and Riehl [13]. Instead, (non-symmetric) colored cyclic operads form a reflective subcategory of the category of cyclic multicategories.

**5.1. Colored trees and colored profiles.** Fix a color set  $\mathfrak{C}$ .

A  $\mathfrak{C}$ -colored tree is a (pinned) tree  $S$  together with a function  $\xi: \text{Ed}(S) \rightarrow \mathfrak{C}$ . If  $\underline{c} = c_0, c_1, \dots, c_n$  is a profile in  $\mathfrak{C}$ , we will write  $\mathbf{tree}(\underline{c}) = \mathbf{tree}^{\mathfrak{C}}(\underline{c})$  for the collection of all  $\mathfrak{C}$ -colored trees  $S$  so that

$$\{0, 1, \dots, n\} \xrightarrow[\cong]{\text{ord}} \text{Legs}(S) \hookrightarrow \text{Ed}(S) \xrightarrow{\xi} \mathfrak{C}$$

takes  $i$  to  $c_i$  for  $0 \leq i \leq n$ . It contains a subset  $\mathbf{rtree}(\underline{c}) = \mathbf{rtree}(c_1, \dots, c_n; c_0)$  if we further insist that  $S$  is a rooted tree (as in definition 1.19).

There are groupoids

- $\Sigma_{\mathfrak{C}}$  whose objects are finite ordered lists  $\underline{c} = c_1, \dots, c_n$  ( $n \geq 0$ ) of elements of  $\mathfrak{C}$ , and morphisms are  $\underline{c}\sigma \xrightarrow{\sigma} \underline{c}$ , where  $\sigma \in \Sigma_m$ , and  $\underline{c}\sigma = (c_{\sigma(1)}, \dots, c_{\sigma(m)})$ .
- $\Sigma_{\mathfrak{C}}^+$  whose objects are finite *non-empty* ordered lists  $\underline{c} = c_0, c_1, \dots, c_n$  ( $n \geq 0$ ) of elements of  $\mathfrak{C}$ , and morphisms are  $\underline{c}\sigma \xrightarrow{\sigma} \underline{c}$ , where  $\sigma \in \Sigma_m^+ := \Sigma_{m+1}$ , and  $\underline{c}\sigma = (c_{\sigma(0)}, c_{\sigma(1)}, \dots, c_{\sigma(m)})$ .

There is a functor  $\mathfrak{C} \times \Sigma_{\mathfrak{C}} \rightarrow \Sigma_{\mathfrak{C}}^+$  which is bijective on objects, and on morphisms by regarding  $\Sigma_m \subset \Sigma_m^+$  as the stabilizer of 0.

Let  $\underline{c} = (c_0, c_1, \dots, c_m)$ ,  $\sigma \in \Sigma_m^+$ , and  $\underline{c}\sigma = (c_{\sigma(0)}, c_{\sigma(1)}, \dots, c_{\sigma(m)})$ . This data determines a morphism  $\underline{c}\sigma \xrightarrow{\sigma} \underline{c}$  (alternatively a map  $\underline{c} \rightarrow \sigma\underline{c} = (c_{\sigma^{-1}(0)}, c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(m)})$ ) in  $\Sigma_{\mathfrak{C}}^+$ . Further, there is a map

$$\begin{aligned} \mathbf{tree}(\underline{c}) &\rightarrow \mathbf{tree}(\underline{c}\sigma) \\ G &\mapsto G\sigma \end{aligned}$$

where  $G\sigma$  has all of the same structure as  $G$  except that the leg ordering  $\text{ord}^{G\sigma}$  is the composite

$$\begin{array}{ccc} & \text{ord}^{G\sigma} & \\ & \curvearrowright & \\ \{0, 1, \dots, m\} & \xrightarrow{\sigma} \{0, 1, \dots, m\} & \xrightarrow{\text{ord}^G} \text{Legs}(G). \end{array}$$

Note that if  $\sigma \in \Sigma_m$ , this same construction gives  $\mathbf{rtree}(\underline{c}; c_0) \rightarrow \mathbf{rtree}(\underline{c}\sigma; c_0)$ .

**Lemma 5.1.** *The above structure induces a contravariant functor*

$$\mathbf{tree}(-) = \mathbf{tree}^{\mathfrak{C}}(-): \Sigma_{\mathfrak{C}}^+ \rightarrow \mathbf{Set}.$$

*There is a subfunctor  $\mathbf{rtree}(-) = \mathbf{rtree}^{\mathfrak{C}}(-): \Sigma_{\mathfrak{C}} \times \mathfrak{C} \rightarrow \mathbf{Set}$ .*

*Proof.* The diagram

$$\begin{array}{ccc} \{0, \dots, m\} & \xrightarrow{\sigma} \{0, \dots, m\} & \\ \tau \uparrow & \searrow \text{ord}^{G\sigma} & \downarrow \text{ord}^G \\ \{0, \dots, m\} & \xrightarrow{\text{ord}^{(G\sigma)\tau}} \text{Legs}(G) & \end{array}$$

shows that  $\text{ord}^{(G\sigma)\tau} = \text{ord}^{G(\sigma\tau)}$ . Thus, the following commutes:

$$\begin{array}{ccc} \mathbf{tree}(\underline{c}) & \xrightarrow{\sigma} \mathbf{tree}(\underline{c}\sigma) & \\ & \searrow \sigma\tau & \downarrow \tau \\ & & \mathbf{tree}(\underline{c}\sigma\tau). \end{array}$$

□

**5.2. Colored cyclic operads.** One can define a  $\mathfrak{C}$ -colored operad in  $(\mathcal{E}, \otimes, 1)$  as an algebra over the monad

$$\begin{aligned} T : \mathcal{E}^{\Sigma_{\mathfrak{C}} \times \mathfrak{C}} &\rightarrow \mathcal{E}^{\Sigma_{\mathfrak{C}} \times \mathfrak{C}} \\ X &\mapsto T(X) \end{aligned}$$

where

$$T(X)(\underline{c}; c_0) = \coprod_{S \in \mathbf{rtree}(c_0; \underline{c})} \bigotimes_{v \in \mathbf{Vt}(S)} X(\xi(\mathbf{in}(v)); \xi(\mathbf{out}(v)))$$

In analogy, we present the following definition of cyclic operad; an analogous definition appears in [33, §5.1] in the monochrome case, and an essentially equivalent definition for colored cyclic operads appears in [25].

**Definition 5.2.** A  $\mathfrak{C}$ -colored cyclic operad in  $(\mathcal{E}, \otimes, 1)$  is an algebra over the monad

$$T^+ : \mathcal{E}^{\Sigma_{\mathfrak{C}}^+} \rightarrow \mathcal{E}^{\Sigma_{\mathfrak{C}}^+}$$

with

$$T^+(X)(\underline{c}) = \coprod_{S \in \mathbf{tree}(\underline{c})} \bigotimes_{v \in \mathbf{Vt}(S)} X(\xi(\mathbf{Nbhd}(v))).$$

We will write  $\mathbf{Cyc}_{\mathfrak{C}}$  for the category of algebras (in  $\mathbf{Set}$ ) over  $T^+$ .

Notice that if  $f : \mathfrak{C} \rightarrow \mathfrak{D}$  is a map of sets, then restriction along  $\Sigma_{\mathfrak{C}}^+ \rightarrow \Sigma_{\mathfrak{D}}^+$  induces a functor  $f^* : \mathbf{Cyc}_{\mathfrak{D}} \rightarrow \mathbf{Cyc}_{\mathfrak{C}}$ .

**Remark 5.3.** Suppose that  $P \in \mathcal{E}^{\Sigma_{\mathfrak{C}}^+}$  and, for  $G \in \mathbf{tree}^{\mathfrak{C}}(\underline{c})$  define

$$P[G] = \bigotimes_{v \in \mathbf{Vt}(G)} P(\xi(\mathbf{Nbhd}(v))).$$

We could instead define a  $\mathfrak{C}$ -colored cyclic operad as an object  $P \in \mathcal{E}^{\Sigma_{\mathfrak{C}}^+}$  together with a collection of composition maps

$$(7) \quad \gamma_P^G : P[G] \rightarrow P(\underline{c})$$

(indexed over  $\mathbf{tree}^{\mathfrak{C}}$ ) which satisfy the appropriate associativity and unitality conditions. We will alternate between these points of view in what follows.

**Definition 5.4.** We will write  $\mathbf{Cyc}$  for the category with

**Objects:** Pairs  $(\mathfrak{C}, P)$ , where  $P \in \mathbf{Cyc}_{\mathfrak{C}}$

**Morphisms:** A map  $(\mathfrak{C}, P) \rightarrow (\mathfrak{D}, Q)$  consists of a pair  $(f, g)$ , where  $f : \mathfrak{C} \rightarrow \mathfrak{D}$  is a map of sets and  $g : P \rightarrow f^*Q$  is a map in  $\mathbf{Cyc}_{\mathfrak{C}}$ .

**Remark 5.5.** Equivalently, we could define a map  $\alpha : (\mathfrak{C}, P) \rightarrow (\mathfrak{D}, Q)$  as a collection consisting of a set map

$$\alpha_0 : \mathfrak{C} \rightarrow \mathfrak{D}$$

as well as, for each  $\underline{c}$  in  $\Sigma_{\mathfrak{C}}^+$ , a map

$$\alpha_{\underline{c}} : P(\underline{c}) \rightarrow Q(\alpha_0 \underline{c})$$

so that certain diagrams involving units and compositions commute.

**Definition 5.6.** If  $P$  is a cyclic operad, define an ordinary operad  $UP$  by setting  $\text{Col}(UP) = \text{Col}(P)$  and

$$(UP)(c_1, \dots, c_n; c_0) = P(c_0, \dots, c_n),$$

with composition defined by the map  $\mathbf{rtree}(\underline{c}; c_0) \rightarrow \mathbf{tree}(c_0, \underline{c})$ . That is, to define  $\gamma_T : (UP)[T] \rightarrow (UP)(\underline{c}; c_0)$ , regard  $T$  as an unrooted tree and let  $\gamma_T$  be the map  $P[T] \rightarrow P(c_0, \underline{c})$ . For  $\alpha : P \rightarrow Q$  in  $\mathbf{Cyc}$ ,  $U\alpha$  is defined by setting  $(U\alpha)_0 = \alpha_0 : \text{Col}(P) \rightarrow \text{Col}(Q)$  and

$$\begin{array}{ccc} (UP)(c_1, \dots, c_n; c_0) & \xrightarrow{U\alpha} & (UQ)(\alpha_0 c_1, \dots, \alpha_0 c_n; \alpha_0 c_0) \\ \downarrow = & & \downarrow = \\ P(c_0, c_1, \dots, c_n) & \xrightarrow{\alpha} & Q(\alpha_0 c_0, \alpha_0 c_1, \dots, \alpha_0 c_n) \end{array}$$

Thus we have defined the forgetful functor  $U : \mathbf{Cyc} \rightarrow \mathbf{Op}$ .

**Lemma 5.7.** *The functor  $U : \mathbf{Cyc} \rightarrow \mathbf{Op}$  is faithful.*

*Proof.* Suppose that  $\alpha, \beta \in \mathbf{Cyc}(P, Q)$  with  $\alpha \neq \beta$ . If  $\alpha_0 \neq \beta_0$ , then  $(U\alpha)_0 \neq (U\beta)_0$ , hence  $U\alpha \neq U\beta$ . If  $\alpha_0 = \beta_0$ , then there exists a profile  $\underline{c} = c_0, c_1, \dots, c_n = c_0, \underline{c}'$  with

$$\alpha_{\underline{c}} \neq \beta_{\underline{c}} : P(\underline{c}) \rightarrow Q(\alpha_0 \underline{c}).$$

But these are the maps

$$(U\alpha)_{\underline{c}'; c_0}, (U\beta)_{\underline{c}'; c_0} : (UP)(\underline{c}'; c_0) \rightarrow (UQ)(\alpha_0 \underline{c}'; \alpha_0 c_0),$$

hence  $U\alpha \neq U\beta$ .  $\square$

The following example gives two cyclic operad structures on the same underlying operad; this is the minimal such example.

**Example 5.8.** Let  $M$  be the (unital) monoid with two non-identity elements  $x$  and  $y$  and  $xy = yx = xx = yy = e$ . Then  $M$  determines a (monochrome) operad  $O$  with

$$O(n) = \begin{cases} * & n = 0 \\ \{x, y, e\} & n = 1 \\ \emptyset & n > 1 \end{cases}$$

with the operadic multiplication given by multiplication in  $M$ . The operad  $O$  admits two distinct cyclic structures  $C$  and  $C'$ . For the first, the action of  $\Sigma_2$  interchanges  $x$  and  $y$ , while in the second, the action fixes  $x$  and fixes  $y$ . These two structures are not isomorphic as cyclic operads because the  $\Sigma_2$ -sets  $C(1) = \Sigma_2 \amalg *$  and  $C'(1) = * \amalg * \amalg *$  are not isomorphic.

## 6. A FUNCTOR FROM $\Xi$ TO THE CATEGORY OF CYCLIC OPERADS

In this section, we will show that given a tree  $R$ , there is a cyclic (colored) operad  $C(R) \in \mathbf{Cyc}$  with  $\text{Col}(C(R)) = \text{Ed}(R)$ . Further, the assignment  $R \mapsto C(R)$  is the object part of a functor  $C : \Xi \rightarrow \mathbf{Cyc}$ . This functor is faithful but not full.

Let  $\text{Prof}(\mathfrak{C})$  be the set of non-empty ordered lists of elements in  $\mathfrak{C}$ ; it is the set of objects of the groupoid  $\Sigma_{\mathfrak{C}}^+$ . There is a forgetful functor  $\mathbf{Cyc}_{\mathfrak{C}} \rightarrow \mathcal{E}^{\Sigma_{\mathfrak{C}}^+} \rightarrow \mathcal{E}^{\text{Prof}(\mathfrak{C})}$ . Write

$$F_{\mathfrak{C}} : \mathcal{E}^{\text{Prof}(\mathfrak{C})} \rightleftarrows \mathbf{Cyc}_{\mathfrak{C}} : U_{\mathfrak{C}}$$

for the corresponding adjunction.

**Definition 6.1.** Let  $S$  be a tree and  $\mathfrak{C} = \text{Ed}(S)$ . Then  $S$  determines an object  $Z = Z^S$  of  $\mathbf{Set}^{\text{Prof}(\mathfrak{C})}$  with

$$(8) \quad Z_{\underline{c}} = \begin{cases} \{v\} & \xi(\text{Nbhd}(v)) = \underline{c} \\ \emptyset & \text{otherwise.} \end{cases}$$

The cyclic operad  $C(S)$  is defined as  $F_{\mathfrak{C}}(Z)$ .

Notice that if  $T \in \Omega$  is a rooted tree and  $F : \mathbf{Op} \rightarrow \mathbf{Cyc}$ , then

$$C(\iota(T)) = F(\Omega(T)).$$

Suppose  $R \in \Xi$ . Write  $\lambda(v) \in C(R)(\xi \text{Nbhd}(v))$  for the image of  $v$  under the unit map  $Z \rightarrow U_{\mathfrak{C}}C(R)$ . If  $G \in \text{Sbgph}(R)$  and  $\underline{c} = (c_0, \dots, c_n)$  is an ordering for  $\text{Legs}(G) \subseteq \text{Ed}(R)$ , then we may regard  $G$  as an element of  $\mathbf{tree}^{\text{Ed}(R)}(\underline{c})$  as follows. We have a composition map

$$\gamma_{C(R)}^G : C(R)[G] \rightarrow C(R)(\underline{c}),$$

and write

$$(9) \quad \overline{G} = \gamma_{C(R)}^G \left( \bigotimes_{v \in \text{Vt}(G)} \lambda(v) \right) \in C(R)(\underline{c})$$

for the element determined by  $G$ .

**Definition 6.2.** We now define a functor  $\Xi \rightarrow \mathbf{Cyc}$ , which, on objects, takes  $S$  to  $C(S)$ . Suppose  $\phi \in \Xi(R, S)$ ; write  $\mathfrak{C} = \text{Col}(C(R)) = \text{Ed}(R)$ . We now define  $C(\phi) : C(R) \rightarrow C(S)$ . On colors, the map is defined to be  $\phi_0 : \text{Ed}(R) \rightarrow \text{Ed}(S)$ . By definition 5.4, defining  $C(\phi)$  comes down to specifying a morphism  $C(R) \rightarrow \phi_0^*C(S)$  in  $\mathbf{Cyc}_{\mathfrak{C}}$ . This map is defined to be the adjoint of

$$v \mapsto \begin{cases} \overline{\phi_1(v)} & \text{if } \phi_0|_{\text{Nbhd}(v)} \text{ is injective} \\ \text{id}_e & \text{if } v \text{ is bivalent and } \phi_0(\text{Nbhd}(v)) = \{e\} \end{cases}$$

which is a map  $Z \rightarrow U_{\mathfrak{C}}\phi_0^*C(S)$  in  $\mathbf{Set}^{\text{Prof}(\mathfrak{C})}$ .

**Proposition 6.3.** *The rule from definition 6.2 actually defines a functor*

$$C : \Xi \rightarrow \mathbf{Cyc}.$$

*Proof.* By construction,  $C(\text{id}_S) = \text{id}_{C(S)}$ .

Suppose that we have two maps  $\phi : R \rightarrow S$  and  $\psi : S \rightarrow T$  of  $\Xi$ . To show that  $C(\psi) \circ C(\phi) = C(\psi \circ \phi)$ , it suffices to check it on vertices  $v \in \text{Vt}(R)$ . We have

$$\begin{aligned}
C(\psi)(C(\phi)(v)) &= C(\psi) \left( \gamma_{C(S)}^{\phi_1(v)} \left[ \bigotimes_{w \in \phi_1(v)} \lambda(w) \right] \right) \\
&= \gamma_{C(T)}^{\psi_0(\phi_1(v))} \left( \bigotimes_{w \in \phi_1(v)} C(\psi)(\lambda(w)) \right) && C(\psi) \text{ a map in } \mathbf{Cyc} \\
&= \gamma_{C(T)}^{\psi_0(\phi_1(v))} \left( \bigotimes_{w \in \phi_1(v)} \gamma_{C(T)}^{\psi_1(w)} \left[ \bigotimes_{t \in \psi_1(w)} \lambda(t) \right] \right) \\
&= \gamma_{C(T)}^H \left( \bigotimes_{w \in \phi_1(v)} \bigotimes_{t \in \psi_1(w)} \lambda(t) \right) && \text{since } C(T) \text{ is a cyclic operad} \\
&= \gamma_{C(T)}^H \left( \bigotimes_{t \in H} \lambda(t) \right)
\end{aligned}$$

where

$$H = \bigcup_{w \in \psi_0(\phi_1(v))} \psi_1(w).$$

But this  $H$  is just  $(\psi\phi)_1(v)$ , hence

$$C(\psi)(C(\phi)(v)) = \gamma_{C(T)}^{(\psi\phi)_1(v)} \left( \bigotimes_{t \in (\psi\phi)_1(v)} \lambda(t) \right) = C(\psi \circ \phi)(v).$$

□

**Theorem 6.4.** *The functor  $C : \Xi \rightarrow \mathbf{Cyc}$  is faithful.*

*Proof.* Let  $\phi, \psi \in \Xi(R, S)$ , and suppose that  $C(\phi) = C(\psi)$ . Write  $f = \phi_0 = \psi_0$  for the common function on color sets. We certainly have  $\phi_1(v) = \psi_1(v)$  whenever  $f|_{\text{Nbhd}(v)}$  is not injective. If  $f|_{\text{Nbhd}(v)}$  is injective, then we know  $\overline{\phi_1(v)} = \overline{\psi_1(v)}$  in  $C(S)(f(\text{Nbhd}(v))) = (f^*C(S))(\text{Nbhd}(v))$ . The set  $C(S)(\underline{d})$  may be identified with the subset of  $\mathbf{tree}(\underline{d})$  consisting of graphs each of whose vertices is labeled by a vertex of  $S$ . But now  $\phi_1(v)$  and  $\psi_1(v)$  are elements of

$$\begin{aligned}
\{G \in \text{Sbgph}(S) \mid \text{Legs}(G) = f(\text{Nbhd}(v)) \text{ as unordered sets}\} &\subset C(S)(f(\text{Nbhd}(v))) \\
&\subset \mathbf{tree}(f(\text{Nbhd}(v)))
\end{aligned}$$

and hence may be identified with  $\overline{\phi_1(v)}$  and  $\overline{\psi_1(v)}$ . Thus  $\phi_1(v) = \psi_1(v)$ . □

**Example 6.5** ( $C : \Xi \rightarrow \mathbf{Cyc}$  is not full). Consider the graph  $L_1 = \text{---}\bullet\text{---}$  with edges 0 and 1 and vertex  $v$ . There are exactly four elements in  $\Xi(L_1, L_1)$ , corresponding to the four maps of edge sets  $\{0, 1\} \rightarrow \{0, 1\}$ . But there are infinitely many maps in  $\text{hom}_{\mathbf{Cyc}}(C(L_1), C(L_1))$ . One example which is not in  $C(\Xi(L_1, L_1))$  is the map  $f : C(L_1) \rightarrow C(L_1)$  which on color sets is  $f(i) = 0$  and on morphisms is specified by  $f(v) = v \circ_1 v$ :

$$\text{---}\overset{0}{\bullet}\overset{1}{\bullet}\text{---}\overset{0}{\bullet}\text{---}.$$



## 7. THE NERVE THEOREM

Define the category of cyclic dendroidal sets to be the presheaf category  $\mathbf{Set}^{\Xi^{op}}$ . Given a tree  $S$ , we will write  $\Xi[S] = \text{hom}_{\Xi}(-, S)$  for the object represented by  $S$ . Another class of examples of cyclic dendroidal sets are the nerves of cyclic operads.

**Definition 7.1** (Nerve). There is a functor  $N : \mathbf{Cyc} \rightarrow \mathbf{Set}^{\Xi^{op}}$  defined, on an object  $O \in \mathbf{Cyc}$  by

$$N(O) = \text{hom}_{\mathbf{Cyc}}(C(-), O) \in \mathbf{Set}^{\Xi^{op}}.$$

The nerve theorem states that  $\mathbf{Cyc}$  embeds fully-faithfully into cyclic dendroidal sets, and that its essential image consists of those objects satisfying a Segal condition. Note that our nerve theorem does not fit into the general monadic framework for nerve theorems from [40, 2], as, for homotopical reasons, we have chosen a non-full subcategory  $\Xi$  of  $\mathbf{Cyc}$  as our indexing category (contrast with the paragraph after [2, Definition 2.3], where the indexing category  $\Theta_T$  is always a full subcategory).

**7.1. Segal cores.** The *Segal core* of  $\Xi[S]$ , denoted  $\text{Sc}[S]$ , is defined to be the union

$$\bigcup_{v \in \text{Vt}(S)} \Xi[\star_v] \subseteq \Xi[S],$$

where  $\star_v$  is regarded as a subobject of  $S$ . Alternatively, given  $S$  define a category  $\mathcal{C}^S$  with<sup>5</sup>  $\text{Ob}(\mathcal{C}^S) = \text{Ed}(S) \amalg \text{Vt}(S)$ , and non-identity maps

$$\{e \rightarrow v \mid e \in \text{Nbhd}(v)\}.$$

There is a functor  $\mathbf{F} : \mathcal{C}^S \rightarrow \mathbf{Set}^{\Xi^{op}}$  which on objects is given by

$$\begin{aligned} \mathbf{F}(e) &= \Xi[\eta] \\ \mathbf{F}(v) &= \Xi[\star_{|v|}]. \end{aligned}$$

For  $0 \leq k \leq n$ , write

$$(10) \quad \begin{aligned} \alpha_k : \eta &\rightarrow \star_n \\ e &\mapsto k \end{aligned}$$

for the map in  $\Xi$  picking out the  $k$ th edge of  $\star_n$ . Define  $\mathbf{F}(\text{ord}^v(k) \rightarrow v)$  to be the inclusion  $(\alpha_k)_* : \Xi[\eta] \rightarrow \Xi[\star_{|v|}]$  which hits  $k \in \Xi[\star_{|v|}]_{\eta} = \{0, 1, \dots, |v| - 1\}$ . Then  $\text{colim}_{\mathcal{C}^S} \mathbf{F} \cong \text{Sc}[S]$ .

Given an object  $X \in \mathbf{Set}^{\Xi^{op}}$ , and  $S \in \Xi$ , there is a map

$$\zeta_S : X_S = \text{hom}(\Xi[S], X) \rightarrow \text{hom}(\text{Sc}[S], X) = \lim_{\mathcal{C}^S} X_{\mathbf{F}}.$$

**Lemma 7.2.** *If  $X = N(O)$ , then for each  $S \in \Xi$ ,*

$$\zeta_S : X_S \rightarrow \lim_{\mathcal{C}^S} X_{\mathbf{F}}$$

*is an isomorphism.*

<sup>5</sup>Note that  $\mathcal{C}^S$  is a subcategory of, but not in general equivalent to,  $(\text{sk}_1 \Xi) \downarrow S$ .

*Proof.* On the left hand side, we have (writing  $\mathfrak{C} = \text{Ed}(S)$  and  $C(S) = F_{\mathfrak{C}}(Z)$ )

$$\begin{aligned}
X_S &= \text{hom}_{\mathbf{Cyc}}(C(S), O) \\
&= \{(f_0, f_1) \mid f_0 \in \mathbf{Set}(\text{Ed}(S), \text{Col}(O)), f_1 \in \text{hom}_{\mathbf{Cyc}_{\mathfrak{C}}}(C(S), f_0^*O)\} \\
&= \{(f_0, f_1) \mid f_0 \in \mathbf{Set}(\mathfrak{C}, \text{Col}(O)), f_1 \in \text{hom}_{\mathbf{Cyc}_{\mathfrak{C}}}(F_{\mathfrak{C}}Z, f_0^*O)\} \\
&= \coprod_{f_0 \in \mathbf{Set}(\mathfrak{C}, \text{Col}(O))} \text{hom}_{\mathbf{Cyc}_{\mathfrak{C}}}(F_{\mathfrak{C}}Z, f_0^*O) \\
&= \coprod_{f_0 \in \mathbf{Set}(\mathfrak{C}, \text{Col}(O))} \text{hom}_{\mathbf{Set}^{\text{Prof}(\mathfrak{C})}}(Z, U_{\mathfrak{C}}f_0^*O) \\
&= \coprod_{f_0 \in \mathbf{Set}(\mathfrak{C}, \text{Col}(O))} \prod_{v \in \text{Vt}(S)} (U_{\mathfrak{C}}f_0^*O)(\text{Nbhd}(v)) \\
&= \coprod_{f_0 \in \mathbf{Set}(\mathfrak{C}, \text{Col}(O))} \prod_{v \in \text{Vt}(S)} O(f_0 \text{Nbhd}(v))
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lim_{\mathcal{C}^S} X_{\mathbf{F}} &\subseteq \prod_{e \in \text{Ed}(S)} X_{\mathbf{F}(e)} \times \prod_{v \in \text{Vt}(S)} X_{\mathbf{F}(v)} \\
&= \prod_{e \in \mathfrak{C}} X_{\eta} \times \prod_{v \in \text{Vt}(S)} X_{\star_{|v|}} \\
&= \prod_{e \in \mathfrak{C}} \text{Col}(O) \times \prod_{v \in \text{Vt}(S)} X_{\star_{|v|}}
\end{aligned}$$

is the subset (see [32, V.2, Theorem 2]) consisting of those sequences  $((x_e)_{e \in \text{Ed}(S)}, (x_v)_{v \in \text{Vt}(S)})$  so that  $\alpha_k^*(x_v) = x_e$  (where  $\text{ord}^v(k) = e$ ). Furthermore, we have

$$\begin{aligned}
X_{\star_{|v|}} &= \text{hom}_{\mathbf{Cyc}}(C(\star_{|v|}), O) \\
&= \prod_{g_0 \in \mathbf{Set}(\{0, 1, \dots, |v|-1\}, \text{Col}(O))} O(g_0(0), g_0(1), \dots, g_0(|v|-1)).
\end{aligned}$$

So now we can regard  $\lim_{\mathcal{C}^S} X_{\mathbf{F}}$  as a subset of

$$\prod_{e \in \mathfrak{C}} \text{Col}(O) \times \prod_{v \in \text{Vt}(S)} \left( \prod_{g_0 \in \mathbf{Set}(\{0, 1, \dots, |v|\}, \text{Col}(O))} O(g_0(0), g_0(1), \dots, g_0(|v|-1)) \right).$$

Using this description, we find

$$\begin{aligned}
\lim_{\mathcal{C}^S} X_{\mathbf{F}} &= \{ \langle (c_e)_{e \in \mathfrak{C}}, (g_0^v, o_v)_{v \in \text{Vt}(S)} \rangle \mid g_0^v((\text{ord}^v)^{-1}(e)) = c_e \} \\
&= \prod_{c \in \mathbf{Set}(\mathfrak{C}, \text{Col}(O))} \prod_{v \in \text{Vt}(S)} O(c(\text{Nbhd}(v))) = X_S.
\end{aligned}$$

□

We now show that the Segal condition identifies those objects in the essential image of the nerve functor.

**Proposition 7.3.** *Let  $X \in \mathbf{Set}^{\Xi^{op}}$ . If the Segal map*

$$\zeta_S : X_S \rightarrow \lim_{\mathcal{C}^S} X_{\mathbf{F}}$$

is an isomorphism for each tree  $S$  with at least one vertex, then there is an  $O \in \mathbf{Cyc}$  with  $X \cong N(O)$ .

*Proof.* We construct the cyclic operad  $O$ , which has color set  $\mathfrak{C} = X_\eta$ . Given a non-empty  $\mathfrak{C}$ -profile  $\underline{c} = c_0, c_1, \dots, c_n$ , we define  $O(c_0, c_1, \dots, c_n)$  as the pullback

$$\begin{array}{ccc} O(\underline{c}) & \longrightarrow & X_{\star_{n+1}} \\ \downarrow & & \downarrow \prod \alpha_k^* \\ \{(c_0, \dots, c_n)\} & \longrightarrow & \prod_{k=0}^n X_\eta \end{array}$$

where  $\alpha_k$  is as in (10). We must now define the compositions

$$\gamma_O^S : O[S] \rightarrow O(\xi(\text{Legs}(S)))$$

for all  $\mathfrak{C}$ -colored trees  $S$ . If  $S = \eta$  with  $\xi(e) = c$ , let  $L_1 \rightarrow \eta$  be the unique map, and define  $\gamma_O^S$  by

$$\begin{array}{ccc} X_\eta & \xleftarrow{\supseteq} & \{c\} \\ \downarrow & & \downarrow \gamma_O^S \\ X_{L_1} & \xleftarrow{\supseteq} & O(c, c). \end{array}$$

If  $S$  contains a vertex, write  $\underline{d} = d_0, \dots, d_m = \xi(\text{Legs}(S))$  and notice that the dashed arrows exist in the below diagram,

$$\begin{array}{ccccc} O[S] & \xrightarrow{=} & \prod_{v \in \text{Vt}(S)} O(c_0^v, \dots, c_{n_v}^v) & \longrightarrow & \prod_{v \in \text{Vt}(S)} X_{\star_{n_v+1}} \\ \downarrow \gamma_O^S & & \downarrow X_S & \xrightarrow{\cong} & \lim_{\mathcal{C}^S} X_{\mathbf{F}} \\ O(\underline{d}) & \xrightarrow{\subseteq} & X_{\star_{m+1}} & & \end{array}$$

where  $\star_{m+1} \rightarrow S$  is the map that takes  $v$  to  $S \in \text{Sbgph}(S)$ . It is straightforward to check that this structure makes  $O$  into an operad, and, further, that  $N(O) \cong X$ .  $\square$

**Theorem 7.4.** *The functor  $N : \mathbf{Cyc} \rightarrow \mathbf{Set}^{\Xi^{op}}$  is a fully-faithful embedding.*

*Proof.* By lemma 5.7, we know that  $U : \mathbf{Cyc} \rightarrow \mathbf{Op}$  is faithful. Further, we know (see [34, Ex. 4.2]) that the dendroidal nerve  $N_d : \mathbf{Op} \rightarrow \mathbf{Set}^{\Omega^{op}}$  is fully-faithful. Hence  $N_d U$  is a faithful functor. We have the following diagram of adjunctions

$$\begin{array}{ccc} \mathbf{Set}^{\Omega^{op}} & \xrightleftharpoons[N_d]{} & \mathbf{Op} \\ \iota_! \uparrow \downarrow \iota^* & & \uparrow \downarrow F \\ \mathbf{Set}^{\Xi^{op}} & \xrightleftharpoons[N]{} & \mathbf{Cyc} \end{array}$$

so, since  $\iota^* N = U N_d$ , we must have  $N$  is faithful as well.

Given a map

$$f : X = N(O) \rightarrow N(P) = Y$$

of cyclic dendroidal sets, we must show there is a map  $\tilde{f} : O \rightarrow P$  of cyclic operads so that  $N(\tilde{f}) = f$ . Write  $\mathfrak{C} = \text{Col}(O)$  and  $\mathfrak{D} = \text{Col}(P)$ . We first have

$$\tilde{f}_0 : \mathfrak{C} = X_\eta \xrightarrow{f} Y_\eta = \mathfrak{D}.$$

To define  $\tilde{f}$ , we must now define a map  $O \rightarrow \tilde{f}_0^* P$  in  $\mathbf{Cyc}_{\mathfrak{C}}$ .

In this paragraph we define  $U_c(O) \rightarrow U_c(\tilde{f}_0^* P)$  in  $\mathbf{Set}^{\Sigma_{\mathfrak{C}}^+}$ . Let  $\underline{c}$  be a profile and  $g : \{0, 1, \dots, n-1\} \rightarrow \mathfrak{C}$  be  $g(i) = c_i$ . Then  $g$  determines a summand inclusion  $O(\underline{c}) \hookrightarrow X_{\star_n}$ . We then have

$$\begin{array}{ccc} O(\underline{c}) & \cdots \cdots \cdots \rightarrow & P(\tilde{f}_0 \underline{c}) \\ \downarrow & & \downarrow \\ X_{\star_n} & \xrightarrow{f} & Y_{\star_n}, \end{array}$$

where the top map exists by compatibility with the maps  $\Xi[\eta] \rightarrow \Xi[\star_n]$ .

It remains to show that this map respects the cyclic operad structure in  $O$  and  $\tilde{f}_0^* P$ . But this follows from commutativity of the diagrams

$$\begin{array}{ccc} X_{\star_{n+m-1}} & \longrightarrow & Y_{\star_{n+m-1}} \\ \downarrow d & & \downarrow d \\ X_{\star_n \vee_{i,j} \star_m} & \longrightarrow & Y_{\star_n \vee_{i,j} \star_m}. \end{array} \quad (\text{Composition})$$

$$\begin{array}{ccc} X_{\star_n} & \longrightarrow & Y_{\star_n} \\ \downarrow \sigma & & \downarrow \sigma \\ X_{\star_n} & \longrightarrow & Y_{\star_n}. \end{array} \quad (\text{Automorphism})$$

$$\begin{array}{ccc} X_\eta & \longrightarrow & Y_\eta \\ \downarrow & & \downarrow \\ X_{\star_n} & \longrightarrow & Y_{\star_n}. \end{array} \quad (\text{Identities})$$

We have thus defined a map  $\tilde{f} : O \rightarrow P$  of cyclic operads. To see that  $N(\tilde{f}) = f$ , notice that  $N(\tilde{f})_{\star_n} = f_{\star_n}$  and  $N(\tilde{f})_\eta = f_\eta$ , then apply lemma 7.2.  $\square$

**Remark 7.5.** Example 6.5, shows, in particular, that the composite

$$\begin{array}{ccc} \Xi & \longrightarrow & \mathbf{Cyc} \longrightarrow \mathbf{Set}^{\Xi^{op}} \\ S & \longmapsto & NC(S) \end{array}$$

is not the Yoneda embedding, since the Yoneda embedding is fully-faithful.

## 8. BERGER-MOERDIJK-REEDY MODEL STRUCTURE

In this section, we recall the Berger-Moerdijk-Reedy model structure on categories of diagrams indexed by a generalized Reedy category (definition 3.1). The basic reference is the paper [3], while many of these results in the case of classical Reedy categories may be found in [23, Chapter 15]. The results of this section are likely well-known among experts, at least in certain special cases.

A model category  $\mathcal{M}$  is called  $\mathbb{R}$ -projective if, for every  $r \in \mathbb{R}$ , the category of (right)  $\text{Aut}(r)$ -equivariant maps  $\mathcal{M}^{\text{Aut}(r)}$  admits the model structure where weak equivalences and fibrations are detected in  $\mathcal{M}$ . This occurs, for instance, if  $\mathcal{M}$  is cofibrantly generated (see [23, 11.6.1]) or if  $\text{Aut}(r) = \{e\}$  for all  $r \in \mathbb{R}$ .

If  $r \in \mathbb{R}$ , let  $\mathbb{R}^+(r)$  be the category whose objects are *non-invertible* maps in  $\mathbb{R}^+$  with codomain  $r$ , and whose morphisms  $\alpha \rightarrow \beta$  are commutative triangles

$$\begin{array}{ccc} s & \xrightarrow{\sigma} & s' \\ & \searrow \alpha & \swarrow \beta \\ & r & \end{array}$$

in  $\mathbb{R}^+$ . This is a full subcategory of  $\mathbb{R}^+ \downarrow r$ . Similarly, for each  $r \in \mathbb{R}$ , there is a full subcategory  $\mathbb{R}^-(r) \subseteq r \downarrow \mathbb{R}^-$  whose objects are non-invertible morphisms with domain  $r$ .

If  $Z \in \mathcal{M}^{\mathbb{R}}$ , define, for each  $r \in \mathbb{R}$ , define *latching* and *matching* objects

$$L_r Z = \text{colim}_{\substack{\mathbb{R}^+(r) \\ \alpha: s \rightarrow r}} Z_s \qquad M_r Z = \lim_{\substack{\mathbb{R}^-(r) \\ \alpha: r \rightarrow s}} Z_s$$

which come equipped with maps

$$L_r Z \rightarrow Z_r \rightarrow M_r Z$$

in  $\mathcal{M}^{\text{Aut}(r)}$ .

**Theorem 8.1** (Theorem 1.6, [3]). *If  $\mathbb{R}$  is a generalized Reedy category and  $\mathcal{M}$  is an  $\mathbb{R}$ -projective model category, then the diagram category  $\mathcal{M}^{\mathbb{R}}$  admits a model structure where  $f : X \rightarrow Y$  is a*

- *weak equivalence if, for each  $r \in \mathbb{R}$ ,  $f_r : X_r \rightarrow Y_r$  is a weak equivalence in  $\mathcal{M}$ ;*
- *cofibration if, for each  $r \in \mathbb{R}$ ,*

$$X_r \amalg_{L_r X} L_r Y \rightarrow Y_r$$

*is a cofibration in  $\mathcal{M}^{\text{Aut}(r)}$ ; and*

- *fibration if, for each  $r \in \mathbb{R}$ ,*

$$X_r \rightarrow M_r X \times_{M_r Y} Y_r$$

*is a fibration in  $\mathcal{M}$ .*

We will call this model structure the *Berger-Moerdijk-Reedy* model structure on  $\mathcal{M}^{\mathbb{R}}$ . It would be convenient to know when this model structure is left proper and cellular, so that we can guarantee the existence of left Bousfield localizations [23, 4.1.1].

**Theorem 8.2.** *If  $\mathbb{R}$  is a generalized Reedy category and  $\mathcal{M}$  is left proper and cofibrantly generated, then the Berger-Moerdijk-Reedy model structure on  $\mathcal{M}^{\mathbb{R}}$  is left proper.*

Before approaching the proof, let us give the following lemma. It explains the ‘cofibrantly generated’ assumption in the theorem statement; a weaker hypothesis<sup>6</sup> on  $\mathcal{M}$  may be sufficient depending on choice of  $\mathbb{R}$ , but we do not know of examples

<sup>6</sup>Namely, that  $\mathcal{M}^{\text{Aut}(r)} \rightarrow \mathcal{M}$  preserves cofibrations for all  $r \in \mathbb{R}$

of model categories  $\mathcal{M}$  which are  $\mathbb{R}$ -projective for a *generic* generalized Reedy category  $\mathbb{R}$  except when  $\mathcal{M}$  is already cofibrantly generated.

**Lemma 8.3.** *If  $\mathbb{R}$  is a generalized Reedy category,  $\mathcal{M}$  is cofibrantly generated, and  $f : A \rightarrow B$  is a Reedy cofibration, then for each  $r \in \mathbb{R}$  the map  $f_r : A_r \rightarrow B_r$  is a cofibration in  $\mathcal{M}$ .*

*Proof.* A variation on [3, 5.3] shows that, for each  $r \in \mathbb{R}$ , the map  $L_r A \rightarrow L_r B$  is a cofibration in  $\mathcal{M}$ . Then  $A_r \rightarrow A_r \amalg_{L_r A} L_r B$  is a pushout of a cofibration

$$\begin{array}{ccc} L_r A & \longrightarrow & A_r \\ \downarrow & & \downarrow \\ L_r B & \longrightarrow & A_r \amalg_{L_r A} L_r B \end{array}$$

hence is a cofibration [23, 7.2.12(a)].

Since  $f$  is a Reedy cofibration,

$$A_r \amalg_{L_r A} L_r B \rightarrow B_r$$

is a cofibration in  $\mathcal{M}^{\text{Aut}(r)}$ . We wish to show that it is also a cofibration in  $\mathcal{M}$ . Since  $\mathcal{M}$  is cofibrantly generated, we can apply the second part of [4, 2.5.1] to the group homomorphism  $i : \{e\} \rightarrow \text{Aut}(r)$  to see that the restriction functor  $\mathcal{M}^{\text{Aut}(r)} \rightarrow \mathcal{M}$  takes cofibrations to cofibrations for all  $r \in \mathbb{R}$ .

Now  $f_r : A_r \rightarrow B_r$  is the composite of two cofibrations in  $\mathcal{M}$

$$A_r \rightarrow A_r \amalg_{L_r A} L_r B \rightarrow B_r,$$

hence is also a cofibration in  $\mathcal{M}$ . □

*Proof of theorem 8.2.* Suppose we have a pushout diagram

$$(11) \quad \begin{array}{ccc} A & \twoheadrightarrow & B \\ \downarrow \simeq & & \downarrow \\ X & \longrightarrow & X \amalg_A B \end{array}$$

in  $\mathcal{M}^{\mathbb{R}}$ ; we wish to show that  $B \rightarrow X \amalg_A B$  is a weak equivalence. Evaluating (11) at  $r \in \mathbb{R}$  gives the pushout square

$$\begin{array}{ccc} A_r & \twoheadrightarrow & B_r \\ \downarrow \simeq & & \downarrow \\ X_r & \longrightarrow & X_r \amalg_{A_r} B_r. \end{array}$$

By lemma 8.3,  $A_r \rightarrow B_r$  is a cofibration in  $\mathcal{M}$ ; since  $\mathcal{M}$  is left proper, this implies that  $B_r \rightarrow X_r \amalg_{A_r} B_r$  is a weak equivalence. Since  $r$  was arbitrary,  $B \rightarrow X \amalg_A B$  is a weak equivalence. □

**Proposition 8.4.** *If  $\mathcal{M}$  is cellular, then so is  $\mathcal{M}^{\mathbb{R}}$ .*

*Proof.* The proof given in [23, 15.7] goes through, with the caveat that in the proof of [23, 15.7.1], one should use lemma 8.3 in place of [23, 15.3.11]. □

**8.1. Reduced presheaves in simplicial sets.** Let  $\mathbb{R}$  be a generalized dualizable Reedy category. We assume that  $\mathbb{R}$  has a unique object  $\eta$  of degree zero and, for any  $r \in \mathbb{R}$ , the set  $\mathbb{R}(r, \eta)$  is either empty or has exactly one element. Examples of such categories include  $\Delta$ ,  $\Omega$ , and  $\Xi$ .

Let  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  be the full subcategory of  $\mathbf{sSet}^{\mathbb{R}^{op}}$  consisting of those  $X$  so that  $X_\eta = \Delta[0]$ . As this category has been thoroughly analyzed in [7] in the case of  $\mathbb{R} = \Delta$  and [9] in the case of  $\mathbb{R} = \Omega$ , many of the arguments and constructions in the remainder of this section should look familiar. Write

$$\mathcal{I} : \mathbf{sSet}_*^{\mathbb{R}^{op}} \hookrightarrow \mathbf{sSet}^{\mathbb{R}^{op}}$$

for the inclusion. This functor admits a left adjoint

$$\mathcal{R} : \mathbf{sSet}^{\mathbb{R}^{op}} \rightarrow \mathbf{sSet}_*^{\mathbb{R}^{op}}$$

which we now describe explicitly.

**Definition 8.5.** We define the *reduction* of an object  $X \in \mathbf{sSet}^{\mathbb{R}^{op}}$ .

- Let  $X^{(0)}$  denote the subobject of  $X$  whose elements factor through  $\eta$ ; that is,

$$X_r^{(0)} = \{\mathbb{R}[r] \xrightarrow{f} \mathbb{R}[\eta] \xrightarrow{t} X\} \subseteq \{\mathbb{R}[r] \xrightarrow{g} X\} = X_r.$$

Notice that

$$X_r^{(0)} \cong \begin{cases} X_\eta & \text{if } \mathbb{R}(r, \eta) = * \\ \emptyset & \text{if } \mathbb{R}(r, \eta) = \emptyset; \end{cases}$$

there is a unique map  $X^{(0)} \rightarrow \mathbb{R}[\eta]$ .

- Define  $\mathcal{R}(X)$  as the pushout

$$\begin{array}{ccc} X^{(0)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbb{R}[\eta] & \longrightarrow & \mathcal{R}(X). \end{array}$$

Suppose  $X \in \mathbf{sSet}_*^{\mathbb{R}^{op}}$ . If there is not a map  $r \rightarrow \eta$ , then  $\mathcal{R}(X)_r = X_r$ . Suppose  $Z \in \mathbf{sSet}_*^{\mathbb{R}^{op}}$ . If there is a map from  $r$  to  $\eta$  then it is unique,  $\mathbb{R}(r, \eta) = \{f\}$ ; this implies that  $Z_r$  has a natural basepoint given by  $f^*(t)$ , where  $t \in Z_\eta$  is the unique element.

The category  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  is bicomplete: Limits and directed colimits are computed in the larger category  $\mathbf{sSet}^{\mathbb{R}^{op}}$ , while finite coproducts are given by the formula

$$(12) \quad (X \amalg Y)_r = \begin{cases} X_r \amalg Y_r & \mathbb{R}(r, \eta) = \emptyset \\ X_r \vee Y_r & \mathbb{R}(r, \eta) = *. \end{cases}$$

Equivalently,  $X \amalg Y = \mathcal{R}(\mathcal{I}(X) \amalg \mathcal{I}(Y))$ .

**Lemma 8.6.** The inclusion functor  $\mathcal{I} : \mathbf{sSet}_*^{\mathbb{R}^{op}} \rightarrow \mathbf{sSet}^{\mathbb{R}^{op}}$  preserves pushouts.

*Proof.* If

$$\begin{array}{ccc} \mathcal{I}X & \longrightarrow & \mathcal{I}Y \\ \downarrow & & \downarrow \\ \mathcal{I}Z & \longrightarrow & A \end{array}$$

is a pushout diagram in  $\mathbf{sSet}^{\mathbb{R}^{op}}$ , then  $A_\eta = \Delta[0]$ . Since  $A$  is already in  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  and  $\mathcal{I}$  is fully-faithful,  $A$  is also the pushout of  $Z \leftarrow X \rightarrow Y$  in  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$ .  $\square$

Given an object  $r \in \mathbb{R}$ , recall that the *boundary* of  $r$  is the subobject  $\partial\mathbb{R}[r] \subseteq \mathbb{R}[r] = \mathbb{R}(-, r)$  so that  $\partial\mathbb{R}[r]_s \subseteq \mathbb{R}(s, r)$  consists of those maps  $s \rightarrow r$  which factor through an object of degree less than  $d(r)$ . In particular, if  $d(s) < d(r)$ ,  $\partial\mathbb{R}[r]_s = \mathbb{R}[r]_s$ .

**Definition 8.7.** The set  $I$  consists of all inclusions

$$(\partial\mathbb{R}[r] \times \Delta[n]) \bigcup_{\partial\mathbb{R}[r] \times \partial\Delta[n]} (\mathbb{R}[r] \times \partial\Delta[n]) \rightarrow \mathbb{R}[r] \times \Delta[n]$$

for  $n \geq 0$  and  $r \in \mathbb{R}$ . The set  $J$  consists of all inclusions

$$(\partial\mathbb{R}[r] \times \Delta[n]) \bigcup_{\partial\mathbb{R}[r] \times \Lambda^k[n]} (\mathbb{R}[r] \times \Lambda^k[n]) \rightarrow \mathbb{R}[r] \times \Delta[n].$$

The set  $I$  (resp.  $J$ ) is a set of generating (acyclic) cofibrations for  $\mathbf{sSet}^{\mathbb{R}^{op}}$ . [3, 7.6]

We now make a further restriction on our generalized Reedy category  $\mathbb{R}$ , which will imply that all (co)domains of elements of  $I \cup J$  are small relative to the whole category.

**Lemma 8.8.** *Suppose that for each  $r \in \mathbb{R}$ , there are only finitely many maps in  $\mathbb{R}^+$  with codomain  $r$ . If  $A \in \mathbf{sSet}^{\mathbb{R}^{op}}$  is a domain or a codomain of a map in  $I \cup J$ , then  $A$  is small. Furthermore, the object  $\mathcal{R}(A) \in \mathbf{sSet}_*^{\mathbb{R}^{op}}$  is small.*

*Proof.* Fix a  $\lambda$ -sequence

$$(13) \quad X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_\beta \rightarrow X_{\beta+1} \rightarrow \cdots \quad (\beta < \lambda)$$

of morphisms of  $\mathbf{sSet}^{\mathbb{R}^{op}}$  and write  $Y = \operatorname{colim}_{\beta < \lambda} X_\beta$ . We wish to show that

$$\operatorname{colim}_{\beta < \lambda} \operatorname{hom}(A, X_\beta) \rightarrow \operatorname{hom}(A, \operatorname{colim}_{\beta < \lambda} X_\beta) = \operatorname{hom}(A, Y)$$

is an isomorphism of sets. Let

$$\varrho_{r,n} = (\mathbb{R} \times \Delta)^+(r, [n]) \subseteq (\mathbb{R}^+ \times \Delta^+) \downarrow (r, [n])$$

be shorthand for the full subcategory whose objects are non-invertible morphisms and let  $\varrho_{r,n}^k$  be the full subcategory of  $\varrho_{r,n}$  whose objects are those  $(s, [m]) \rightarrow (r, [n])$  so that if  $s \rightarrow r$  is an isomorphism then  $[m] \rightarrow [n]$  is not  $d^k$  (or  $\operatorname{id}_{[n]}$ ). If

$$A = (\partial\mathbb{R}[r] \times \Delta[n]) \bigcup_{\partial\mathbb{R}[r] \times \partial\Delta[n]} (\mathbb{R}[r] \times \partial\Delta[n]),$$

set  $\varrho = \varrho_{r,n}$ , and if

$$A = (\partial\mathbb{R}[r] \times \Delta[n]) \bigcup_{\partial\mathbb{R}[r] \times \Lambda^k[n]} (\mathbb{R}[r] \times \Lambda^k[n]),$$

set  $\varrho = \varrho_{r,n}^k$ . Otherwise, if  $A = \mathbb{R}[r] \times \Delta[n]$ , let  $\varrho = (\mathbb{R}^+ \times \Delta^+) \downarrow (r, [n])$ . We then have

$$A = \operatorname{colim}_{\substack{(s, [m]) \rightarrow (r, [n]) \\ \in \varrho}} \mathbb{R}[s] \times \Delta[m],$$



and so

$$\begin{aligned}
 \text{colim}_{\beta < \lambda} [\text{hom}(A, X_\beta)] &= \text{colim}_{\beta < \lambda} \left[ \text{hom} \left( \text{colim}_{\varrho} \mathbb{R}[s] \times \Delta[m], X_\beta \right) \right] \\
 &= \text{colim}_{\beta < \lambda} \left[ \lim_{\varrho} \left( \text{hom}(\mathbb{R}[s] \times \Delta[m], X_\beta) \right) \right] \\
 (14) \quad &= \text{colim}_{\beta < \lambda} \left[ \lim_{\varrho} \left( (X_\beta)_{s,m} \right) \right] = \lim_{\varrho} \left[ \text{colim}_{\beta < \lambda} \left( (X_\beta)_{s,m} \right) \right] \\
 &= \lim_{\varrho} \left[ \left( \text{colim}_{\beta < \lambda} X_\beta \right)_{s,m} \right] = \lim_{\varrho} (Y_{s,m}) \\
 &= \lim_{\varrho} [\text{hom}(\mathbb{R}[s] \times \Delta[m], Y)] \\
 &= \text{hom} \left( \text{colim}_{\varrho} \mathbb{R}[s] \times \Delta[m], Y \right) = \text{hom}(A, Y).
 \end{aligned}$$

The only interesting step is interchanging the limit and the colimit in line (14), which is valid since  $\lambda$  is directed and  $\varrho$  is finite [11, 2.13.4].

To show that  $\mathcal{R}(A)$  is small, let (13) be a  $\lambda$ -sequence in  $\mathbf{sSet}^{\mathbb{R}^{op}}_*$ . By adjointness and the fact that  $\mathcal{I}$  commutes with directed colimits, we have the commutative square

$$\begin{array}{ccc}
 \text{colim}_{\beta < \lambda} \text{hom}(\mathcal{R}(A), X_\beta) & \longrightarrow & \text{hom} \left( \mathcal{R}(A), \text{colim}_{\beta < \lambda} X_\beta \right) \\
 \downarrow & & \parallel \\
 & = & \text{hom} \left( A, \mathcal{I} \text{colim}_{\beta < \lambda} X_\beta \right) \\
 & & \parallel \\
 \text{colim}_{\beta < \lambda} \text{hom}(A, \mathcal{I} X_\beta) & \xrightarrow{\cong} & \text{hom} \left( A, \text{colim}_{\beta < \lambda} \mathcal{I} X_\beta \right).
 \end{array}$$

□

Notice the following.

**Lemma 8.9.** *Let  $A$  be a domain or codomain of an element of  $I \cup J$ , with  $A \subset \mathbb{R}[r] \times \Delta[n]$ . If  $d(r) > 0$ , then  $A^{(0)} = \mathbb{R}[r]^{(0)} \times \Delta[n]$ . If  $r = \eta$ , then  $A^{(0)} = A$ , hence  $\mathcal{R}(A) = \mathbb{R}[\eta]$ .*

*Proof.* The important point is that

$$(\partial \mathbb{R}[r])^{(0)} = \begin{cases} \mathbb{R}[r]^{(0)} & d(r) > 0 \\ \emptyset & d(r) = 0. \end{cases}$$

If  $d(r) > 0$ , then

$$\mathbb{R}[r]^{(0)} \times \Delta[n] = (\partial \mathbb{R}[r] \times \Delta[n])^{(0)} \subset A^{(0)} \subset (\mathbb{R}[r] \times \Delta[n])^{(0)} = \mathbb{R}[r]^{(0)} \times \Delta[n].$$

If  $r = \eta$ , then  $A$  is one of  $\mathbb{R}[\eta] \times \partial \Delta[n]$ ,  $\mathbb{R}[\eta] \times \Lambda^k[n]$ , or  $\mathbb{R}[\eta] \times \Delta[n]$ , hence  $A^{(0)} = A$ . □

**Proposition 8.10.** *Suppose that  $j : A \rightarrow \mathbb{R}[r] \times \Delta[n]$  is in  $I$  or  $J$  and  $d(r) > 0$ . If  $j \in I$ , then  $\mathcal{R}(j)$  is a cofibration in  $\mathbf{sSet}^{\mathbb{R}^{op}}$ , while if  $j \in J$ , then  $\mathcal{R}(j)$  is an acyclic cofibration in  $\mathbf{sSet}^{\mathbb{R}^{op}}$ .*

*Proof.* The cube with front and back squares pushouts

$$\begin{array}{ccccc}
 A^{(0)} & \xrightarrow{\quad} & A & \searrow & \\
 \downarrow & \searrow^{8.9} & \downarrow & & \downarrow \\
 & (\mathbb{R}[r] \times \Delta[n])^{(0)} & \xrightarrow{\quad} & \mathbb{R}[r] \times \Delta[n] & \\
 \downarrow & \downarrow & \downarrow & & \downarrow \\
 \mathbb{R}[\eta] & \xrightarrow{\quad} & \mathcal{R}(A) & \searrow & \\
 \downarrow & \searrow^= & \downarrow & & \downarrow \\
 & \mathbb{R}[\eta] & \xrightarrow{\quad} & \mathcal{R}(\mathbb{R}[r] \times \Delta[n]) & 
 \end{array}$$

reduces to a rectangle

$$\begin{array}{ccccc}
 A^{(0)} & \longrightarrow & A & \longrightarrow & \mathbb{R}[r] \times \Delta[n] \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbb{R}[\eta] & \longrightarrow & \mathcal{R}(A) & \longrightarrow & \mathcal{R}(\mathbb{R}[r] \times \Delta[n])
 \end{array}$$

where the big rectangle and the left square are pushouts, hence so is the right square.

The result now follows since the class of (acyclic) cofibrations is closed under pushouts.  $\square$

**Proposition 8.11.** *Suppose that  $j : A \rightarrow \mathbb{R}[\eta] \times \Delta[n]$  is in  $I$  or  $J$ . Then  $\mathcal{R}(j)$  is the identity map on  $\mathbb{R}[\eta]$ .*

*Proof.* This is a direct consequence of the second part of lemma 8.9 and the fact that  $\text{hom}(\mathbb{R}[\eta], \mathbb{R}[\eta]) = \mathbb{R}(\eta, \eta) = *$ .  $\square$

**Lemma 8.12.** *If  $j \in J$ , then  $\mathcal{R}(j)_s$  is an acyclic cofibration in  $\mathbf{sSet}$  for every  $s \in \mathbb{R}$ . If  $j \in I$ , then  $\mathcal{R}(j)_s$  is a cofibration in  $\mathbf{sSet}$  for every  $s \in \mathbb{R}$ .*

*Proof.* By the previous two propositions, we know that  $\mathcal{R}(j)$  is an (acyclic) cofibration in the Berger-Moerdijk-Reedy model structure on  $\mathbf{sSet}^{\mathbb{R}^{op}}$ . The result then follows from lemma 8.3.  $\square$

**Theorem 8.13.** *Suppose that for each  $r \in \mathbb{R}$ , there are only finitely many maps in  $\mathbb{R}^+$  with codomain  $r$ . Then the Berger-Moerdijk-Reedy model structure lifts along the adjunction*

$$\mathcal{R} : \mathbf{sSet}^{\mathbb{R}^{op}} \rightleftarrows \mathbf{sSet}_*^{\mathbb{R}^{op}} : \mathcal{I}.$$

*In other words,  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  admits a cofibrantly-generated model structure with weak equivalences (resp. fibrations) those maps which are weak equivalences (resp. fibrations) in the larger category  $\mathbf{sSet}^{\mathbb{R}^{op}}$ .*

*Proof.* We apply Kan's lifting theorem [23, 11.3.2]. We know that  $\mathcal{R}I$  and  $\mathcal{R}J$  satisfy the small object argument by lemma 8.8, hence condition (1) is satisfied. For condition (2) we know that  $\mathcal{R}(j)_s$  is an acyclic cofibration in  $\mathbf{sSet}$  for any  $s \in \mathbb{R}$  by lemma 8.12. Then given any relative  $\mathcal{R}J$ -cell complex  $X \rightarrow Y$  in  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$ , we also have  $X_s \rightarrow Y_s$  is an acyclic cofibration for each  $s \in \mathbb{R}$ . Thus  $\mathcal{I}X \rightarrow \mathcal{I}Y$  is a weak equivalence, and condition (2) is established.  $\square$

Notice that the inclusion functor does not admit a right adjoint (as it does not preserve finite coproducts), hence cannot be a left Quillen functor. Nevertheless, we have the following, which hinges on the fact that  $\mathcal{I}$  preserves pushouts and nonempty filtered colimits.

**Proposition 8.14.** *The inclusion functor  $\mathcal{I} : \mathbf{sSet}_*^{\mathbb{R}^{op}} \rightarrow \mathbf{sSet}^{\mathbb{R}^{op}}$  preserves (acyclic) cofibrations.*

*Proof.* The inclusion functor preserves all weak equivalences, so it is enough to show that it preserves cofibrations. By propositions 8.10 and 8.11, we know that  $\mathcal{I}$  takes generating cofibrations in  $\mathcal{RI}$  to cofibrations in  $\mathbf{sSet}^{\mathbb{R}^{op}}$ . The inclusion functor preserves pushouts (lemma 8.6) and transfinite composition, so takes relative  $\mathcal{RI}$ -cell complexes to cofibrations. Since every cofibration in  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  is a retract of a relative  $\mathcal{RI}$ -cell complex, every cofibration in  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  is again a cofibration in  $\mathbf{sSet}^{\mathbb{R}^{op}}$ .  $\square$

**Corollary 8.15.** *An map  $X \rightarrow Y$  in  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  is a fibration if and only if it is a fibration in  $\mathbf{sSet}^{\mathbb{R}^{op}}$ . In particular,  $X$  is fibrant in  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  if and only if  $\mathcal{I}(X)$  is fibrant in  $\mathbf{sSet}^{\mathbb{R}^{op}}$ .*

*Proof.* The forward implication is clear since  $\mathcal{I}$  is a right Quillen functor. For the reverse, suppose that  $\mathcal{I}(X) \rightarrow \mathcal{I}(Y)$  is a fibration and we are given a diagram

$$(15) \quad \begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

of  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  with  $A \rightarrow B$  an acyclic cofibration. By proposition 8.14,  $\mathcal{I}(A) \rightarrow \mathcal{I}(B)$  is again an acyclic cofibration, hence there exists a lift  $\mathcal{I}(B) \rightarrow \mathcal{I}(X)$  in the diagram

$$\begin{array}{ccc} \mathcal{I}(A) & \longrightarrow & \mathcal{I}(X) \\ \downarrow & \nearrow \alpha & \downarrow \\ \mathcal{I}(B) & \longrightarrow & \mathcal{I}(Y) \end{array}$$

Since  $\mathcal{I}$  is fully-faithful,  $\alpha$  gives a lift in the original diagram (15). Thus  $X \rightarrow Y$  is a fibration.  $\square$

The following has antecedents elsewhere in special cases, for instance in the proof of [9, 4.3].

**Proposition 8.16.** *With the hypotheses of theorem 8.13, the model structure on  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  is left proper and cellular.*

*Proof.* For left properness, it is enough to show that the pushout of a weak equivalence along a generating cofibration  $\mathcal{R}(j) \in \mathcal{RI}$  is again a weak equivalence; this follows from lemma 8.12, left properness of  $\mathbf{sSet}$ , and that weak equivalences are levelwise weak equivalences.

We now turn to cellularity, and aim to verify the three conditions from [23, 12.1.1]. We know that (2) holds by the second part of lemma 8.8. Recall from proposition 8.4 that  $\mathbf{sSet}^{\mathbb{R}^{op}}$  is a cellular model category. Cofibrations in  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$

are also cofibrations (proposition 8.14) in the cellular model category  $\mathbf{sSet}^{\mathbb{R}^{op}}$ , hence are effective monomorphisms. Thus (3) holds.

It remains to show (1). All elements of the form  $\mathbb{R}[r] \times \Delta[n]$  are compact relative to  $I$  since they are codomains of elements of  $I$  and  $\mathbf{sSet}^{\mathbb{R}^{op}}$  is cellular. If  $A$  is a domain or a codomain of an element  $j \in I \cup J$ , then  $\mathcal{R}(A)$  is compact (relative to  $I$ ) by [23, 10.8.8], using that  $A$ ,  $A^{(0)}$  and  $\mathbb{R}[\eta] = \mathbb{R}[\eta] \times \Delta[0]$  may be written as colimits of the basic objects  $\mathbb{R}[r] \times \Delta[n]$ . This shows that the set  $\mathcal{R}(I)$  is a set of cofibrations of  $\mathbf{sSet}^{\mathbb{R}^{op}}$  with compact domains. By [23, 11.4.9], if  $W \in \mathbf{sSet}^{\mathbb{R}^{op}}$  is compact relative to  $I$ , then  $W$  is compact relative to  $\mathcal{R}(I)$ . In particular,  $\mathcal{R}(A)$  is compact relative to  $\mathcal{R}(I)$ . Presented relative  $\mathcal{R}(I)$ -cell complexes in  $\mathbf{sSet}^{\mathbb{R}^{op}}$  with  $X_0 \in \mathbf{sSet}_*^{\mathbb{R}^{op}}$  are the same thing as presented relative  $\mathcal{R}(I)$ -cell complexes in  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$ , hence all domains and codomains of elements in  $\mathcal{R}(I) \cup \mathcal{R}(J)$  are compact in  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$ . Thus [23, 12.1.1(1)] holds, and we conclude that  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  is a cellular model category.  $\square$

**8.2. Simplicial model structures.** As in the case of an ordinary Reedy category, the Berger-Moerdijk-Reedy model structure on  $\mathbf{sSet}^{\mathbb{R}^{op}}$  is a *simplicial model structure* (see [23, Ch. 9]). The structure is given as follows:

**Definition 8.17.** Suppose that  $X, Y \in \mathbf{sSet}^{\mathbb{R}^{op}}$  and  $K \in \mathbf{sSet}$ .

- The object  $X \otimes K$  is defined on objects by

$$(X \otimes K)_r = X_r \times K \in \mathbf{sSet}.$$

- The object  $Y^K$  is defined on objects by

$$(Y^K)_r = \text{map}_{\mathbf{sSet}}(K, Y_r) \in \mathbf{sSet}$$

where  $\text{map}_{\mathbf{sSet}}$  is the simplicial mapping space, that is,  $\text{map}_{\mathbf{sSet}}(A, B)_n = \text{hom}_{\mathbf{sSet}}(A \times \Delta[n], B)$ .

- The mapping spaces  $\text{map}(X, Y)$  are defined in degree  $n$  by

$$\text{map}(X, Y)_n = \text{hom}(X \otimes \Delta[n], Y).$$

The simplicially-enriched category  $\mathbf{sSet}^{\mathbb{R}^{op}}$  satisfies the two conditions to be a simplicial model category [23, 9.1.6], namely

**M6:** For every two objects  $X, Y$  and every  $K \in \mathbf{sSet}$

$$\text{map}(X \otimes K, Y) \cong \text{map}_{\mathbf{sSet}}(K, \text{map}(X, Y)) \cong \text{map}(X, Y^K)$$

(with isomorphisms natural in the three variables).

**M7:** If  $A \rightarrow B$  is a cofibration and  $X \rightarrow Y$  is a fibration, then

$$(16) \quad \text{map}(B, X) \rightarrow \text{map}(A, X) \times_{\text{map}(A, Y)} \text{map}(B, Y)$$

is a fibration of simplicial sets. If either map is a weak equivalence, then (16) is also a weak equivalence.

Any full subcategory of a simplicially-enriched category is simplicially-enriched; we will now work towards theorem 8.23, where we show that  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  is a simplicial model category.

**Notation 8.18.** For the remainder of this section, we will write

$$\mathcal{M} = \mathbf{sSet}^{\mathbb{R}^{op}} \quad \mathcal{N} = \mathbf{sSet}_*^{\mathbb{R}^{op}}$$

for these two model categories. For  $X, Y \in \mathcal{M}$ ,  $K \in \mathbf{sSet}$ , we will write  $X \otimes_{\mathcal{M}} K := X \otimes K$  and  $\text{map}_{\mathcal{M}}(X, Y) := \text{map}(X, Y)$ .

Recall that we have a Quillen adjunction

$$\mathcal{R} : \mathcal{M} \rightleftarrows \mathcal{N} : \mathcal{I}.$$

**Definition 8.19.** Suppose that  $X, Y \in \mathcal{N} = \mathbf{sSet}_*^{\mathbb{R}^{op}}$  and  $K \in \mathbf{sSet}$ .

- The object  $X \otimes_{\mathcal{N}} K$  is defined to be  $X \otimes_{\mathcal{N}} K = \mathcal{R}(\mathcal{I}(X) \otimes_{\mathcal{M}} K)$ .
- The object  $Y^K$  is defined as  $Y^K = \mathcal{R}[(\mathcal{I}Y)^K]$ .
- We define  $\mathrm{map}_{\mathcal{N}}(X, Y) = \mathrm{map}_{\mathcal{M}}(\mathcal{I}X, \mathcal{I}Y)$ .

**Remark 8.20.** The object  $Z = (\mathcal{I}Y)^K$  already has  $Z_{\eta} = \Delta[0]$ , which explains why we've elected not to distinguish between the exponential in the two categories. Indeed,

$$Z_{\eta} = ((\mathcal{I}Y)^K)_{\eta} = \mathrm{map}_{\mathbf{sSet}}(K, Y_{\eta}) = \mathrm{map}_{\mathbf{sSet}}(K, \Delta[0]) = \Delta[0].$$

**Lemma 8.21.** *There is an isomorphism*

$$\mathrm{map}_{\mathcal{N}}(\mathcal{R}(-), -) \cong \mathrm{map}_{\mathcal{M}}(-, \mathcal{I}(-))$$

*of bifunctors  $\mathcal{N} \times \mathcal{M} \rightarrow \mathbf{sSet}$ .*

*Proof.* Let  $Z \in \mathcal{N}$ ,  $Y \in \mathcal{M}$ , and  $n \geq 0$ . We have

$$\begin{aligned} \mathrm{map}_{\mathcal{N}}(\mathcal{R}(Z), Y)_n &= \mathrm{map}_{\mathcal{M}}(\mathcal{I}\mathcal{R}(Z), \mathcal{I}Y)_n \\ &= \mathrm{hom}_{\mathcal{M}}(\mathcal{I}\mathcal{R}(Z) \otimes_{\mathcal{M}} \Delta[n], \mathcal{I}Y) \\ &= \mathrm{hom}_{\mathcal{M}}(\mathcal{I}\mathcal{R}(Z), (\mathcal{I}Y)^{\Delta[n]}) \\ &= \mathrm{hom}_{\mathcal{M}}(\mathcal{I}\mathcal{R}(Z), i_{\mathcal{R}}(\mathcal{I}Y)^{\Delta[n]}) && \text{Remark 8.20} \\ &= \mathrm{hom}_{\mathcal{N}}(\mathcal{R}(Z), \mathcal{R}(\mathcal{I}Y)^{\Delta[n]}) \\ &= \mathrm{hom}_{\mathcal{M}}(Z, \mathcal{I}\mathcal{R}(\mathcal{I}Y)^{\Delta[n]}) \\ &= \mathrm{hom}_{\mathcal{M}}(Z, (\mathcal{I}Y)^{\Delta[n]}) = \mathrm{map}_{\mathcal{M}}(Z, \mathcal{I}Y)_n. \end{aligned}$$

with all isomorphisms natural in  $Z, Y$ , and  $n$ .  $\square$

**Lemma 8.22.** *With the structure from definition 8.19,  $\mathcal{N} = \mathbf{sSet}_*^{\mathbb{R}^{op}}$  satisfies M6.*

*Proof.* Let  $X, Y \in \mathcal{N}$  and  $K \in \mathbf{sSet}$ . First,

$$\mathrm{map}_{\mathcal{N}}(X \otimes_{\mathcal{N}} K, Y) = \mathrm{map}_{\mathcal{N}}(\mathcal{R}(\mathcal{I}(X) \otimes_{\mathcal{M}} K), Y) = \mathrm{map}_{\mathcal{M}}(\mathcal{I}(X) \otimes_{\mathcal{M}} K, \mathcal{I}Y)$$

by lemma 8.21. Thus, using M6 for  $\mathcal{M}$ , the simplicial set  $\mathrm{map}_{\mathcal{N}}(X \otimes_{\mathcal{N}} K, Y)$  is isomorphic to, on the one hand,

$$\mathrm{map}_{\mathcal{M}}(\mathcal{I}(X), (\mathcal{I}Y)^K) = \mathrm{map}_{\mathcal{M}}(\mathcal{I}X, \mathcal{I}\mathcal{R}(\mathcal{I}Y)^K) = \mathrm{map}_{\mathcal{N}}(X, Y^K)$$

and on the other to

$$\mathrm{map}_{\mathbf{sSet}}(K, \mathrm{map}_{\mathcal{M}}(\mathcal{I}X, \mathcal{I}Y)) = \mathrm{map}_{\mathbf{sSet}}(K, \mathrm{map}_{\mathcal{N}}(X, Y)).$$

$\square$

**Theorem 8.23.** *With the structure from definition 8.19,  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  is a simplicial model category.*

*Proof.* We already saw that M6 holds, so we merely need to check M7. Suppose that  $A \rightarrow B$  is a cofibration in  $\mathcal{N}$  and  $X \rightarrow Y$  is a fibration in  $\mathcal{N}$ . By definition, the map (16)

$$\mathrm{map}_{\mathcal{N}}(B, X) \rightarrow \mathrm{map}_{\mathcal{N}}(A, X) \times_{\mathrm{map}_{\mathcal{N}}(A, Y)} \mathrm{map}_{\mathcal{N}}(B, Y)$$

is equal to

$$\mathrm{map}_{\mathcal{M}}(\mathcal{J}B, \mathcal{J}X) \rightarrow \mathrm{map}_{\mathcal{M}}(\mathcal{J}A, \mathcal{J}X) \times_{\mathrm{map}_{\mathcal{M}}(\mathcal{J}A, \mathcal{J}Y)} \mathrm{map}_{\mathcal{M}}(\mathcal{J}B, \mathcal{J}Y),$$

which is a fibration since  $\mathcal{J}A \rightarrow \mathcal{J}B$  is a cofibration by proposition 8.14 and  $\mathcal{J}X \rightarrow \mathcal{J}Y$  is a fibration since  $\mathcal{J}$  is a right Quillen functor. Since  $\mathcal{J}$  detects weak equivalences, this map is a weak equivalence if one of  $A \rightarrow B$  or  $X \rightarrow Y$  is.  $\square$

## 9. SEGAL CYCLIC OPERADS

In this section we define Segal cyclic operads as certain fibrant objects in  $\mathbf{sSet}_*^{\Xi^{op}}$  which satisfy a Segal condition (definition 9.9). The Segal cyclic objects may be identified as the fibrant objects after we have localized the model structure on  $\mathbf{sSet}_*^{\Xi^{op}}$  with respect to Segal core inclusions. Evidence that this is a valid model for higher cyclic operads is given in propositions 9.8 and 9.10.

We begin this section by specializing the work of section 8 to the cases  $\mathbb{R} = \Omega$  or  $\Xi$ , which leads us immediately to four separate model structures. We show that they fit into a diagram (18) of Quillen adjunctions. Afterward, we discuss the left Bousfield localization, and show that after localization we still have a diagram of Quillen adjunctions. Finally, we check in proposition 9.10 that we are getting something legitimately new, and speculate on the possibility of rigidification theorems.

**Proposition 9.1.** *Using the Berger-Moerdijk-Reedy model structure, the adjunction*

$$\iota_! : \mathbf{sSet}^{\Omega^{op}} \rightleftarrows \mathbf{sSet}^{\Xi^{op}} : \iota^*$$

*is a Quillen adjunction.*

*Proof.* The map  $\iota^*$  preserves weak equivalences since those are defined levelwise, hence it is enough to show that  $\iota^*$  preserves fibrations. If  $T$  is a rooted tree, we will show that

$$(17) \quad \begin{array}{ccc} (\Omega^{op})^-(T) & \longrightarrow & (\Xi^{op})^-(\iota T) \\ \parallel & & \parallel \\ (\Omega^+(T))^{op} & & (\Xi^+(\iota T))^{op} \end{array}$$

is an initial functor (see [32, §IX.3]). This implies that the natural map

$$M_{\iota T} X = \lim_{(\Xi^{op})^-(\iota T)} X_S \rightarrow \lim_{(\Omega^{op})^-(T)} (\iota^* X)_{T'} = M_T(\iota^* X)$$

is an isomorphism. Hence if  $X \rightarrow Y$  is a map in  $\mathbf{sSet}^{\Xi^{op}}$  and  $T \in \Omega$ , the map on the right of the commutative diagram

$$\begin{array}{ccc} (\iota^* X)_T & \longrightarrow & M_T(\iota^* X) \times_{M_T(\iota^* Y)} (\iota^* Y)_T \\ \parallel & & \uparrow \\ X_{\iota T} & \longrightarrow & M_{\iota T}(X) \times_{M_{\iota T}(Y)} Y_{\iota T} \end{array}$$

is an isomorphism. In particular, if  $X \rightarrow Y$  is a fibration, then so is  $\iota^*(X \rightarrow Y)$ .

As promised, we will now show that (17) is an initial functor. To distinguish it from  $\iota$ , we will write  $F$  for this functor:

$$F : (\Omega^+(T))^{op} \rightarrow (\Xi^+(\iota T))^{op}.$$

Suppose that  $S \xrightarrow{\phi} \iota T$  is an object of  $(\Xi^+(\iota T))^{op}$ , that is,  $\phi$  is an element of  $\Xi^+(S, \iota T) \setminus \text{Iso}(\Xi)$ . Our goal is to show that  $F \downarrow \phi$  is nonempty and connected. Write  $s_0 = \odot_{t_0}(\phi)$ , where  $t_0$  is the root of  $T$ ; we have a morphism

$$\mathcal{L}_{t_0}(\phi) : \mathcal{T}(S, s_0) \rightarrow \mathcal{T}(\iota T, t_0) = T.$$

Using the structure map  $f : \iota \mathcal{T}(S, s_0) \xrightarrow{\cong} S$  from definition 2.1, the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{f^{-1}} & \iota \mathcal{T}(S, s_0) \\ & \searrow \phi & \swarrow \iota \mathcal{L}_{t_0}(\phi) \\ & \iota T & \end{array}$$

in  $\Xi$  constitutes an object

$$F(\mathcal{L}_{t_0}(\phi)) \xrightarrow{(f^{-1})^{op}} \phi$$

in  $F \downarrow \phi$ . To show that this category is connected, suppose that we have an arbitrary object

$$F(R \xrightarrow{\alpha} T) \xrightarrow{\gamma^{op}} \phi$$

of  $F \downarrow \phi$ , where  $\alpha^{op} \in (\Omega^{op})^-(T)$ . Such an object corresponds to a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\gamma} & \iota R \\ & \searrow \phi & \swarrow \iota \alpha \\ & \iota T & \end{array}$$

with  $\gamma \in \Xi^+$ . Lifting all maps to  $\Omega$ , we have the diagram

$$\begin{array}{ccccc} \mathcal{T}(S, s_0) & \xrightarrow{\mathcal{L}_{r_0}(\gamma)} & \mathcal{T}(\iota R, r_0) & \xlongequal{\quad} & R \\ & \searrow \mathcal{L}_{t_0}(\phi) & \swarrow \mathcal{L}_{t_0}(\iota \alpha) & & \swarrow \alpha \\ & \mathcal{T}(\iota T, t_0) & \xlongequal{\quad} & T & \end{array}$$

in  $\Omega$ , which commutes by proposition 2.11. Commutativity of the diagram

$$\begin{array}{ccccc} & & \gamma & & \\ & \searrow f^{-1} & & \searrow \iota(\mathcal{L}_{r_0}(\gamma)) & \\ S & \xrightarrow{\quad} & \iota \mathcal{T}(S, s_0) & \xrightarrow{\quad} & \iota R \\ & \searrow \phi & \downarrow F(\mathcal{L}_{t_0}(\phi)) & \swarrow F(\alpha) & \\ & \iota T & & & \end{array}$$

in  $\Xi$  shows that  $\iota(\mathcal{L}_{r_0}(\gamma))$  constitutes a morphism  $\gamma^{op} \rightarrow (f^{-1})^{op}$  in  $F \downarrow \phi$ ; thus this category is connected. Since  $\phi$  was arbitrary, we have that  $F$  is an initial functor.  $\square$

**Corollary 9.2.** *The adjunction*

$$\iota_! : \mathbf{sSet}_*^{\Omega^{op}} \rightleftarrows \mathbf{sSet}_*^{\Xi^{op}} : \iota^*$$

*is a Quillen adjunction.*

*Proof.* We have a commutative diagram

$$\begin{array}{ccc} \mathbf{sSet}^{\Omega^{op}} & \xleftarrow{\iota^*} & \mathbf{sSet}^{\Xi^{op}} \\ \mathcal{J} \uparrow & & \uparrow \mathcal{J} \\ \mathbf{sSet}_*^{\Omega^{op}} & \xleftarrow{\iota^*} & \mathbf{sSet}_*^{\Xi^{op}} \end{array}$$

of right adjoints where all but the bottom map  $\iota^*$  are known to be right Quillen functors. Suppose that  $X \rightarrow Y$  is an (acyclic) fibration in  $\mathbf{sSet}_*^{\Xi^{op}}$ , which implies  $\mathcal{J}\iota^*(X \rightarrow Y) = \iota^*\mathcal{J}(X \rightarrow Y)$  is an (acyclic) fibration in  $\mathbf{sSet}^{\Omega^{op}}$ . Since  $\mathcal{J} : \mathbf{sSet}_*^{\Omega^{op}} \rightarrow \mathbf{sSet}^{\Omega^{op}}$  detects fibrations and weak equivalences we know that  $\iota^*(X \rightarrow Y)$  is an (acyclic) fibration, implying  $\iota^* : \mathbf{sSet}_*^{\Xi^{op}} \rightarrow \mathbf{sSet}_*^{\Omega^{op}}$  is a right Quillen functor.  $\square$

We now have a diagram of Quillen adjunctions

$$(18) \quad \begin{array}{ccc} \mathbf{sSet}^{\Omega^{op}} & \xrightleftharpoons[\iota^*]{\iota_!} & \mathbf{sSet}^{\Xi^{op}} \\ \mathcal{R} \downarrow \uparrow \mathcal{J} & & \downarrow \uparrow \mathcal{J} \\ \mathbf{sSet}_*^{\Omega^{op}} & \xrightleftharpoons[\iota^*]{\iota_!} & \mathbf{sSet}_*^{\Xi^{op}} \end{array}$$

and we wish to localize each of these model structures.

**9.1. Localizations.** Roughly speaking, a left localization of a model category  $\mathcal{M}$  at a set of maps  $C$  is a left Quillen functor from  $\mathcal{M}$  which is initial among all left Quillen functors which take elements of  $C$  to weak equivalences. Recall from [23, 3.1.4] that an object  $W$  of  $\mathcal{M}$  is called  $C$ -local if  $W$  is fibrant and for each  $f : A \rightarrow B$  which is an element of  $C$ , the map  $\mathrm{map}^h(B, W) \rightarrow \mathrm{map}^h(A, W)$  is a weak equivalence of simplicial sets. A map  $g : X \rightarrow Y$  is called a  $C$ -local equivalence if  $\mathrm{map}^h(Y, W) \rightarrow \mathrm{map}^h(X, W)$  is a weak equivalence for every  $C$ -local  $W$ . The left Bousfield localization of  $\mathcal{M}$  with respect to  $C$  [23, 3.3.1], denoted  $\mathcal{L}_C\mathcal{M}$ , is then a model structure on  $\mathcal{M}$  with weak equivalences the  $C$ -local equivalences and cofibrations the ordinary cofibrations in  $\mathcal{M}$ . The fibrant objects in this model structure (if it exists) are precisely the  $C$ -local objects, and the identity functor  $\mathcal{M} \rightarrow \mathcal{L}_C\mathcal{M}$  is a left localization of  $\mathcal{M}$ .

In order to show that the diagram (18) remains a diagram of Quillen adjunctions after localization, we will apply the following lemma several times.

**Lemma 9.3.** *Let  $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$  be a Quillen adjunction. Suppose that  $C \subset \mathcal{M}$  and  $D \subset \mathcal{N}$  are sets of maps with the domain and codomain of each element of  $C$  cofibrant. Suppose further that the left Bousfield localizations  $\mathcal{L}_C\mathcal{M}$  and  $\mathcal{L}_D\mathcal{N}$  exist. If, for each  $c \in C$ , the map  $L(c) \in \mathcal{N}$  is isomorphic to some  $d \in D$ , then*

$$L : \mathcal{L}_C\mathcal{M} \rightleftarrows \mathcal{L}_D\mathcal{N} : R$$

*is a Quillen adjunction.*

*Proof.* There is a left Quillen functor

$$F : \mathcal{M} \rightarrow \mathcal{N} \rightarrow \mathcal{L}_D\mathcal{N}.$$

If  $c \in C$ , then  $c$  is a cofibrant approximation to itself. Further,  $F(c) \cong d \in D$  is a weak equivalence, hence  $F$  takes any cofibrant approximation of  $c$  to a weak



equivalence by [23, 8.1.24(1)]. By [23, 3.3.18(1)], the functor  $F$  is then a left Quillen functor when regarded as a functor  $\mathcal{L}_C\mathcal{M} \rightarrow \mathcal{L}_D\mathcal{N}$ .  $\square$

Let  $\mathbb{R}$  be either  $\Omega$  or  $\Xi$ . Define  $\mathcal{S}_{\mathbb{R}}$  to be the set of Segal core inclusions

$$\mathcal{S}_{\mathbb{R}} = \{\mathrm{Sc}[r] \rightarrow \mathbb{R}[r]\}_{r \neq \eta}$$

**Remark 9.4.** According to §8.2,  $\mathbf{sSet}^{\mathbb{R}^{op}}$  and  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  are *simplicial model categories*. Thus, if  $A$  is cofibrant and  $X$  is fibrant, we may use the simplicial mapping space  $\mathrm{map}(A, X)$  as a model for the homotopy function complex  $\mathrm{map}^h(A, X)$  (by, for example, [23, Example 17.2.4]). Since  $\mathrm{Sc}[r]$ ,  $\mathbb{R}[r]$ ,  $\mathcal{R}(\mathrm{Sc}[r])$ , and  $\mathcal{R}(\mathbb{R}[r])$  are all cofibrant, it suffices to work with  $\mathrm{map}$  rather than  $\mathrm{map}^h$  when discussing  $\mathcal{S}_{\mathbb{R}}$  or  $\mathcal{R}(\mathcal{S}_{\mathbb{R}})$  locality.

Since  $\mathbf{sSet}^{\mathbb{R}^{op}}$  is left proper and cellular by theorem 8.2 and proposition 8.4, the left Bousfield localization

$$\mathcal{L}_{\mathcal{S}_{\mathbb{R}}} \mathbf{sSet}^{\mathbb{R}^{op}}$$

exists by [23, 4.1.1]. Since  $\mathbf{sSet}_*^{\mathbb{R}^{op}}$  is left proper and cellular by proposition 8.16, we can take the left Bousfield localization with respect to the set of maps  $\mathcal{R}(\mathcal{S}_{\mathbb{R}})$ . We call the resulting model structure

$$\mathcal{L}_{\mathcal{R}(\mathcal{S}_{\mathbb{R}})} \mathbf{sSet}_*^{\mathbb{R}^{op}}.$$

For  $\mathbb{R} = \Omega$ , these two model structures appear in [15, Definition 5.4] and [9, Proposition 4.3], respectively.

**Proposition 9.5.** *Let  $\mathbb{R}$  be either  $\Omega$  or  $\Xi$ . Then the Quillen adjunction*

$$\mathcal{R}: \mathbf{sSet}^{\mathbb{R}^{op}} \rightleftarrows \mathbf{sSet}_*^{\mathbb{R}^{op}}: \mathcal{I}$$

*induces a Quillen adjunction*

$$\mathcal{R}: \mathcal{L}_{\mathcal{S}_{\mathbb{R}}} \mathbf{sSet}^{\mathbb{R}^{op}} \rightleftarrows \mathcal{L}_{\mathcal{R}(\mathcal{S}_{\mathbb{R}})} \mathbf{sSet}_*^{\mathbb{R}^{op}}: \mathcal{I}$$

*after taking left Bousfield localization.*

*Proof.* If  $r \neq \eta$  is an object of  $\mathbb{R}$ , then both  $\mathrm{Sc}[r]$  and  $\mathbb{R}[r]$  are cofibrant. Thus lemma 9.3 applies.  $\square$

**Lemma 9.6.** *Suppose  $T \in \Omega$  is a rooted tree with at least one vertex. Then*

$$\iota_!(\mathrm{Sc}[T] \rightarrow \Omega[T]) \cong (\mathrm{Sc}[\iota T] \rightarrow \Xi[\iota T]).$$

*Proof.* Notice that  $\iota_!\Omega[T] \cong \Xi[\iota T]$  since for any  $X \in \mathbf{sSet}^{\Xi^{op}}$ , they are the same on  $\mathrm{hom}(-, X)$ :

$$\mathrm{hom}(\iota_!\Omega[T], X) = \mathrm{hom}(\Omega[T], \iota^*X) = (\iota^*X)_T = X_{\iota T} = \mathrm{hom}(\Xi[T], X).$$

In particular,  $\iota_!(\Omega[C_v]) = \Xi[\star_v]$  and  $\iota_!(\Omega[\eta]) = \Xi[\eta]$ . Since  $\iota_!$  is a left adjoint, it commutes with colimits, and we have

$$\iota_!(\mathrm{Sc}[T] \rightarrow \Omega[T]) \cong (\mathrm{Sc}[\iota T] \rightarrow \Xi[\iota T])$$

using the description of  $\mathrm{Sc}[T]$  from section 7.1.  $\square$

**Corollary 9.7.** *Suppose  $T \in \Omega$  is a rooted tree with at least one vertex. Then*

$$\iota_!(\mathcal{R}(\mathrm{Sc}[T]) \rightarrow \mathcal{R}(\Omega[T])) \cong (\mathcal{R}(\mathrm{Sc}[\iota T]) \rightarrow \mathcal{R}(\Xi[\iota T])).$$

*Proof.* We have

$$\begin{aligned} \iota_! \mathcal{R}(\mathrm{Sc}[T] \rightarrow \Omega[T]) &= \mathcal{R}\iota_!(\mathrm{Sc}[T] \rightarrow \Omega[T]) \\ &\cong \mathcal{R}(\mathrm{Sc}[\iota T] \rightarrow \Xi[\iota T]) \end{aligned} \quad \begin{array}{l} (18) \\ \text{Lemma 9.6.} \end{array}$$

□

**Proposition 9.8.** *The diagram (18) actually gives a diagram*

$$\begin{array}{ccc} \mathcal{L}_{\mathcal{S}_\Omega} \mathbf{sSet}^{\Omega^{op}} & \xrightleftharpoons[\iota^*]{\iota_!} & \mathcal{L}_{\mathcal{S}_\Xi} \mathbf{sSet}^{\Xi^{op}} \\ \mathcal{R} \downarrow \uparrow \mathcal{J} & & \mathcal{R} \downarrow \uparrow \mathcal{J} \\ \mathcal{L}_{\mathcal{R}(\mathcal{S}_\Omega)} \mathbf{sSet}_*^{\Omega^{op}} & \xrightleftharpoons[\iota^*]{\iota_!} & \mathcal{L}_{\mathcal{R}(\mathcal{S}_\Xi)} \mathbf{sSet}_*^{\Xi^{op}} \end{array}$$

of Quillen adjunctions after localizing.

*Proof.* In light of proposition 9.5, we only need to show that the horizontal adjunctions are Quillen adjunctions. The objects  $\mathrm{Sc}[T]$  and  $\Omega[T]$  are cofibrant in  $\mathbf{sSet}^{\Omega^{op}}$ , so the top adjunction is a Quillen adjunction by lemmas 9.3 and 9.6. Since  $\mathcal{R}$  is a left Quillen functor,  $\mathcal{R}(\mathrm{Sc}[T])$  and  $\mathcal{R}(\Omega[T])$  are cofibrant in  $\mathbf{sSet}_*^{\Omega^{op}}$ . Thus the bottom adjunction is a Quillen adjunction by lemma 9.3 and corollary 9.7. □

**Definition 9.9.** A *Segal cyclic operad* is a fibrant object in the model category  $\mathcal{L}_{\mathcal{R}(\mathcal{S}_\Xi)} \mathbf{sSet}_*^{\Xi^{op}}$ .

Notice that every Segal cyclic operad has an underlying Segal operad via the functor  $\iota^*$ .

**Proposition 9.10.** *The Quillen adjunction*

$$\iota_! : \mathcal{L}_{\mathcal{R}(\mathcal{S}_\Omega)} \mathbf{sSet}_*^{\Omega^{op}} \rightleftarrows \mathcal{L}_{\mathcal{R}(\mathcal{S}_\Xi)} \mathbf{sSet}_*^{\Xi^{op}} : \iota^*$$

does not induce an equivalence of homotopy categories. In particular, this adjunction is not a Quillen equivalence.

*Proof.* Consider the two cyclic operads  $C, C'$  in  $\mathbf{Set}$  from example 5.8; recall from that example that  $UC = UC'$ . Let  $X = N(C)$  and  $X' = N(C')$ , and note that  $\iota^* X = \iota^* X'$ . Additionally, let  $A$  be the operad with

$$A(n) = \begin{cases} * & n = 0 \\ \{e, x \mid x^2 = e\} & n = 1 \\ \emptyset & n > 1. \end{cases}$$

The operad  $A$  admits a unique cyclic operad structure where the  $\Sigma_2$  action on  $A(1)$  fixes  $x$ . There is exactly one map of cyclic operads  $A \rightarrow C$ , while there are three maps  $A \rightarrow C'$ ; we will show that this remains true once we pass to the homotopy category of  $\mathcal{L}_{\mathcal{R}(\mathcal{S}_\Xi)} \mathbf{sSet}_*^{\Xi^{op}}$ . Write  $W = N(A)$ , and note that  $W_S = \emptyset$  if  $S$  is non-linear, while  $W_{L_m}$  is the set of words of length  $m$  in the alphabet  $e, x$ . Each of the objects  $W, X, X'$  are fibrant in  $\mathcal{L}_{\mathcal{R}(\mathcal{S}_\Xi)} \mathbf{sSet}_*^{\Xi^{op}}$  by corollary 8.15, lemma 7.2, and the fact that discrete simplicial sets are fibrant.

A cofibrant resolution of  $W$  in the unlocalized model structure  $\mathbf{sSet}_*^{\Xi^{op}}$  is given by  $\widetilde{W} = W \otimes E\Sigma_2 \rightarrow W$ , where  $E\Sigma_2 \rightarrow \Delta[0]$  is a cofibrant resolution of the terminal

object of  $\mathbf{sSet}^{\Sigma_2}$  (where the tensor product is the one from definition 8.19). To see that  $\widetilde{W} \simeq W$ , notice that at  $L_m$  we have

$$\widetilde{W}_{L_m} = \left( (W_{L_m} \setminus \{e^{\times m}\}) \times E\Sigma_2 \right)_+ \simeq \left( (W_{L_m} \setminus \{e^{\times m}\}) \right)_+ = W_{L_m}$$

and  $\widetilde{W}_S = \emptyset$  if  $S$  is nonlinear.

To see that  $\widetilde{W}$  is cofibrant, notice that  $W$  and  $\widetilde{W}$  admit a filtration with  $W^{(k)}$  consisting of words which have  $x$  appearing at most  $k$  times and  $\widetilde{W}^{(k)} = W^{(k)} \otimes E\Sigma_2$ . Then  $\widetilde{W}^{(k)}$  is the pushout in  $\mathbf{sSet}_*^{\Xi^{op}}$ ,

$$\begin{array}{ccc} \mathcal{R}(\partial\Xi[L_k]) \otimes E\Sigma_2 & \longrightarrow & \widetilde{W}^{(k-1)} \\ \downarrow & & \downarrow \\ \mathcal{R}(\Xi[L_k]) \otimes E\Sigma_2 & \longrightarrow & \widetilde{W}^{(k)} \end{array}$$

which implies that  $\widetilde{W}^{(k-1)} \rightarrow \widetilde{W}^{(k)}$  is a cofibration in  $\mathbf{sSet}_*^{\Xi^{op}}$ . Since  $\widetilde{W}^{(0)} = W^{(0)} = \Xi[\eta]$  is the initial object of this category, it follows that  $\widetilde{W} = \text{colim } \widetilde{W}^{(k)}$  is cofibrant.

We now have, using theorem 7.4,

$$\text{hom}(\widetilde{W}, X) = \text{hom}(NA \otimes E\Sigma_2, NC) = \text{hom}(NA, NC^{E\Sigma_2}) = \text{hom}(NA, NC) = \text{hom}(A, C)$$

and similarly  $\text{hom}(\widetilde{W}, X') = \text{hom}(A, C')$ . But these sets are especially easy to understand. Maps of cyclic operads from  $A$  to any other cyclic operad are determined by where we send  $x$ , and we have

$$\begin{aligned} \text{hom}(\widetilde{W}, X) &= \text{hom}(A, C) = \{f_e\} \\ \text{hom}(\widetilde{W}, X') &= \text{hom}(A, C') = \{f_e, f_x, f_y\} \end{aligned}$$

where  $f_a(x) = a$ .

Since  $\widetilde{W}$  is cofibrant in the unlocalized model structure and the objects  $X, X'$  are fibrant in the localized model structure, we can compute the homotopy classes of maps from  $\widetilde{W}$  to  $X$  (or  $X'$ ) in either the unlocalized or localized model structure and will get the same answer by [23, 3.5.2]. We now work in the unlocalized model structure  $\mathbf{sSet}_*^{\Xi^{op}}$ . Assume that there is a left homotopy between  $f_a, f_b : \widetilde{W} \rightarrow W'$ , that is, a diagram

$$\begin{array}{ccc} \widetilde{W} \amalg \widetilde{W} & \xrightarrow{i_0 \amalg i_1} & \text{Cyl}(\widetilde{W}) \xrightarrow[p \simeq]{} \widetilde{W} \\ & & \downarrow H \\ & & X' \end{array}$$

with  $Hi_0 = f_a$ ,  $Hi_1 = f_b$ , and  $pi_0 = \text{id}_{\widetilde{W}} = pi_1$ . Evaluating the diagram at  $L_1$  and taking path components of the resulting spaces, we have

$$\begin{array}{ccc} \pi_0(\widetilde{W}_{L_1}) \amalg \pi_0(\widetilde{W}_{L_1}) & \xrightarrow{i_0 \amalg i_1} & \pi_0(\text{Cyl}(\widetilde{W})_{L_1}) \xrightarrow[p \simeq]{} \pi_0(\widetilde{W}_{L_1}) \\ & & \downarrow H \\ & & \pi_0(X'_{L_1}) \end{array}$$

and  $\pi_0(\widetilde{W}_1) = \{e, x\}$ ,  $\pi_0(X'_1) = \{e, x, y\}$ , and  $i_0 = p^{-1} = i_1$ . Thus  $a = Hi_0(x) = Hp^{-1}(x) = Hi_1(x) = b$  implies that  $a = b$ .

Thus, the three maps in  $\text{hom}(\widetilde{W}, X')$  are homotopically distinct, so  $|\pi(\widetilde{W}, X')| = 3 > 1 = |\pi(\widetilde{W}, X)|$  (see [23, §7.5] for notation). Thus  $X$  and  $X'$  are not isomorphic in the homotopy category of  $\mathcal{L}_{\mathcal{R}(\mathcal{S}_{\Xi})}\mathbf{sSet}_{*}^{\Xi^{op}}$ .  $\square$

There is a model structure on the category of (monochrome) simplicial cyclic operads where the weak equivalences and fibrations are defined as those maps which are levelwise weak equivalences. This follows by considering either the multi-sorted algebraic theory of cyclic operads or the colored operad controlling cyclic operads (for the latter, see [31, §1.6.4]), and then applying [6, Theorem 4.7] or [5, Theorem 2.1].

**Conjecture 9.11.** The model structure for simplicial cyclic operads is Quillen equivalent to  $\mathcal{L}_{\mathcal{R}(\mathcal{S}_{\Xi})}\mathbf{sSet}_{*}^{\Xi^{op}}$ .

Analogous results for simplicial monoids and for simplicial operads appear in [7] and [9], respectively.

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